

Tax Evasion, Public Debt and Aggregate Instability (Very Preliminary Version)

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Abstract

In this article, we develop an endogenous growth model to analyze the relation between tax evasion and public debt accumulation. Our results are threefold. First, our model exhibits a multiplicity of equilibria in the long run: there is a low-growth and high public debt balanced growth path (BGP) and a high-growth and low-public debt BGP. Second, we show the existence of threshold effects in the tax evasion-public debt nexus. In low-growth economies, tax evasion negatively affects public debt while the relation between the two variables is characterized by a U-shaped curve in high-growth economies. Finally, regarding the local stability of the BGPs, we show that the high BGP is always well-determined. However, the topological behavior of the low BGP is more complex: it can either be locally determined, undetermined or overdetermined. In the latter case, a Hopf bifurcation appears depending on the level of tax evasion.

Keywords: Tax evasion, public debt accumulation, endogenous growth, multiple equilibria, Hopf bifurcation

1. Introduction

Tax evasion is one of the major public issues that most countries around the world are facing today. In 2013, the European Commission President Jose-Manuel Barroso said in a speech at the European Parliament that around 1 trillion euros is evaded annually in the EU member states. The United States are up against a similar problem. According to Rogoff (2017), tax evasion is estimated to be more than 3%

of the US GDP every year. In Africa, a report of Global Financial Integrity (2013) has highlighted that tax evasion causes losses estimated at hundreds of millions of dollars. These observations raise a crucial question: what are the macroeconomic consequences of tax evasion in terms of public finance and growth?

Different works have tried to analyze the optimal taxation policies in economies where tax evasion is widespread. Thus, Chen (2003) examines how tax evasion affects the optimal tax rate in a an AK endogenous growth model with productive public expenditures and a balanced-budget rule (hereafter BBR). Similarly to Barro (1990) and Futagami *et al.* (1993), his model reproduces the inverted U-shaped curve between optimal taxation and growth and shows that the optimal optimal tax rate is higher as tax evasion becomes more widespread. According to Chen (2003), an increased tax rate allows compensating for the losses caused by tax evasion. However, this result holds only when the government has no other instruments to finance public spending¹. If we relax the BBR's assumption, the losses caused by tax evasion might also be financed by public borrowing.

Surprisingly, to the best of our knowledge, there is no work that addresses the relation between tax evasion and public debt accumulation. Yet, Litina and Palivos (2013) have highlighted that tax evasion is a key part of the “Greek tragedy”. Similarly, Pappa *et al.* (2015) show, in a DSGE model calibrated using Italian data, that tax evasion leads to substantial losses in output and welfare and amplifies the need to increase the tax rate in order to reduce debt. They conclude that their conclusions hold for countries like Greece, Spain and Portugal as well. Thus, tax evasion seems to strongly contribute to increase fiscal deficits in many European countries. However, the relation may be more complex because tax evasion improves the efficiency of the private sector and and may generate a complex interaction with economic growth. Strategic complementarities and multiple equilibria may emerge (Mauro, 2004 ; Aidt *et al.*, 2008) and their implications in terms of public debt have not yet been explored

¹At the monetary policy level, Roubini and Sala-i-Martin (1995) have suggested to repress the financial sector in order to finance public spending by resorting to seigniorage. However, this strategy is rather difficult to implement in most industrialized countries where the financial sector is highly developed and seigniorage revenues represent a tiny fraction of the overall government's revenues.

in an endogenous growth setup.

The purpose of this paper is to fill this gap in the literature by developing a theoretical framework that allows assessing the relation between tax evasion and public debt accumulation. To this end, we build an endogenous growth model with productive public expenditures where public debt is introduced by relaxing the BBR assumption, in line with the experience of most countries that face positive deficit rates. Contrary to Minea and Villieu (2012) and Menuet et al. (2017) who have modeled similar mechanisms, we assume that households receive a risk premium that positively depends on tax evasion. In a first step, tax evasion is modeled by an exogenous fraction of the government's revenues that households evade in order to increase their disposable income. As a second step, we endogenize tax evasion and consider the optimizing behavior of households who make an effort to evade as much taxes as possible.

Our results are threefold. First, in the steady-state, our model exhibits multiplicity of equilibria. Contrary to Mauro (2004), this multiplicity comes from the interaction between the intertemporal households' saving behavior and the government budget constraint. On the one hand, public debt negatively depends on economic growth because growth reduces the debt burden in the long run. On the other hand, the government budget constraint leads to an inverted U-shaped curve between growth and public debt, due to the dual effect that growth exerts on the marginal productivity of private capital (positive) and the risk premium (negative). This interaction generates a multiplicity of equilibria that leads to two steady-state solutions, i.e. a high-growth and low-public debt solution and a low-growth and high-public debt solution.

Second, regarding the long run consequences of tax evasion on growth and public debt, we show that the low BGP positively depends on tax evasion while the effects of tax evasion on the high BGP are characterized by an inverted U-shaped relation. By contrast, the relation between tax evasion and public debt accumulation is negative in low-growth economies and characterized by a U shaped curve in high growth economies. In high-growth economies, tax evasion exerts a dual effect on public debt. On the one hand, it increases the marginal productivity of private capital by improving the efficiency of the private sector leading to (i) a higher growth and (ii) a

lower debt. On the other hand, tax evasion, by reducing tax revenues, (i) decreases growth and (ii) amplifies the need to resort to public borrowing to finance public spending.

Third, regarding the transition path, our model exhibits complex dynamics. While the high BGP is always saddle-path stable, the determinacy of the low BGP crucially depends on the elasticity of the risk premium. The low BGP can either be locally determined (saddle-path stable), overdetermined (unstable) or undetermined (stable). In the latter case, a Hopf bifurcation occurs, leading to the emergence of limit-cycles. We also show that the behavior of the government towards tax evasion is an essential variable for the determinacy of the low BGP.

The remainder of the paper is structured as follows. Section 2 presents the baseline model. In section 3, we focus on the steady-state properties. Section 4 analyzes the dynamics of the model outside the steady-state. Section 5 proposes an extension of the model to the case where tax evasion is endogenous. Section 6 concludes.

2. The model

We consider a continuous-time endogenous growth model in a closed economy. The economy is populated by two perfectly rational agents: a private sector and a government.

2.1. The private sector

The private sector is represented by an infinitely-lived representative agent who produces and consumes a unique final good. His objective is to maximize the present value of the discounted sum of instantaneous utility functions based on consumption ($c_t > 0$).

$$\mathcal{U}(c_t) = \int_0^{\infty} \exp(-\rho t) u(c_t) dt \tag{1}$$

where $\rho \in (0, \infty)$ corresponds to the subjective discount rate.

To get an endogenous growth path in the long run, we assume a CES utility function where σ corresponds to the intertemporal elasticity of substitution

$$u(c_t) = \begin{cases} \frac{c_t^{1-\sigma} - 1}{1-\sigma} & \text{if } \sigma \neq 1 \\ \log(c_t) & \text{if } \sigma = 1 \end{cases} \quad (2)$$

Moreover, for $\mathcal{U}(c_t)$ to be bounded², we need to ensure that the no-Ponzi game constraint is satisfied, i.e. $(1 - \sigma)\gamma_c < \rho$.³

The production function (y_t), based on physical capital (k_t) and productive public expenditures (g_t), is the same as in Barro (1990). It is described by the following relation :

$$y_t = Ak_t^{1-\alpha}g_t^\alpha \quad (3)$$

where A is a scale parameter and $0 < \alpha < 1$ is the elasticity of output to productive public expenditures (similarly, $0 < 1 - \alpha < 1$ is the elasticity of output to domestic private capital). All variables are per capita. For the sake of simplicity and without any loss of generality, population is normalized to unity.

In this model, households accumulate two assets: public debt securities b_t and private capital k_t . For simplicity, we abstract from capital depreciation. Therefore, the household's instantaneous budget constraint can be expressed as:⁴

$$\dot{k}_t + \dot{b}_t = [1 - \mathcal{P}(\cdot)] R_t b_t + y_t^d - c_t \quad (4)$$

Households use their income (y_t) to consume c_t , to accumulate capital k_t and save

²This condition is necessary to find an optimum for welfare maximization.

³We note γ_x the growth rate of the variable x .

⁴A dot over a variable corresponds to the first derivative of this variable with respect to time: $\dot{x}_t \equiv \frac{\partial x}{\partial t}$.

a part of their revenue in the form of government bonds (b_t). Government bonds are public debt securities and have a return rate noted R_t . Notice that the return rate of bonds (R_t) is different from the real interest rate (r_t). This latter is subject to a risk premium noted $\mathcal{P}(\cdot)$, to be defined below. Finally, y_t^d corresponds to the disposable income of households expressed as

$$y_t^d = [1 - (1 - \eta)\tau]y_t \quad (5)$$

where τ represents the flat-tax rate fixed by the government and η is a parameter denoting tax evasion. Indeed, we consider in this model that households may have incentives to evade taxes in order to increase their disposable income. As Huang and Wei (2006), Dimakou (2013) and Dimakou (2015) among others, we model tax evasion by a simple exogenous parameter reducing tax revenues for the government (see next subsection). However, we also consider that this reduction of the tax revenues collected by the government leads to a proportional increase in the disposable income of households at the aggregate level. Thus, for $\eta \rightarrow 1$, tax evasion is widespread in the economy. Conversely, tax evasion is very low when $\eta \rightarrow 0$.

2.2. The government

The government determines the tax rate and borrows from the household in order to provide productive public expenditures. However, the government faces tax evasion. Following Huang and Wei (2006), Dimakou (2013), Dimakou (2015), we model tax evasion affects by a parameter η reducing tax revenues. Hence, the government budget constraint is expressed as

$$\dot{b}_t = R_t b_t + g_t - (1 - \eta)\tau y_t. \quad (6)$$

The expression (6) constitutes an extension of the Barro's (1990) budget constraint for two reasons. *First*, it allows productive public expenditures to be financed by public borrowing. *Second*, the government faces a tax leakage caused by tax evasion.

In addition, we assume the risk premium as a positive function of the level of tax evasion. The higher the level of tax evasion, the higher the risk premium. Thus, we define the expression of the risk premium as follows :

$$\mathcal{P}(\cdot) = 1 - \left[\frac{(1 - \eta)\tau\bar{y}_t}{\bar{b}_t} \right]^\varepsilon, \quad (7)$$

where ε denotes the sensitivity of the risk premium and \bar{b}_t and \bar{y}_t are respectively the equilibrium values of public debt and output. Notice that the representative household takes \bar{b}_t and \bar{y}_t as given values in his maximization program.

Finally, in order to obtain an endogenous growth solution, productive public expenditures must be endogenous in the government budget constraint and must therefore converge on some constant in the long run. To characterize this fact, we assume that the government follows the following fiscal rule

$$\dot{b}_t = \theta y_t, \quad (8)$$

where θ is a constant target of deficit.

2.3. Equilibrium

The equilibrium of the model is obtained by solving the household's program. This amounts to maximizing (1) subject to the constraints (2), (3), (4) and (5). The resolution of the household's maximization program is provided in Appendix A. It leads to the usual Keynes-Ramsey rule governing the law of motion of consumption

$$\gamma_c := \frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\sigma}. \quad (9)$$

The goods market equilibrium provides the growth rate of capital

$$\gamma_k := \frac{\dot{k}_t}{k_t} = y_k - g_k - c_k. \quad (10)$$

where $g_k = g_t/k_t$ and $c_k = c_t/k_t$. The GDP-to-capital ratio is expressed as

$$y_k := \frac{y_t}{k_t} = Ag_k^\alpha, \quad (11)$$

The law of motion of public debt stems from the fiscal rule followed by the government (8)

$$\gamma_b := \frac{\dot{b}_t}{b_t} = \frac{\theta y_k}{b_k}, \quad (12)$$

and b_k is obtained by using the government budget constraint.

$$\theta y_k = R_t b_k + g_k - (1 - \eta)\tau y_k. \quad (13)$$

In addition, the tradeoff between public debt accumulation and private capital accumulation is given by the following relation

$$R_t = \frac{r_t}{1 - \mathcal{P}(\cdot)}, \quad (14)$$

where $R_t > r_t \forall t$ and the real interest rate r_t is equal to the marginal productivity of capital such that $r_t = [1 - (1 - \eta)\tau](1 - \alpha)y_k$. Notice that the return of bonds is positively related to the risk premium. The higher the risk premium, the higher the return of bonds.

Finally, replacing R_t by its expression in (14), we obtain the expression of the equilibrium public debt-to-capital ratio which only depends on the productive public expenditures-to-capital ratio

$$b_k = y_k \left\{ \frac{[(\theta + (1 - \eta)\tau)y_k - g_k] [(1 - \eta)\tau]^\varepsilon}{(1 - \alpha)[1 - (1 - \eta)\tau]y_k^2} \right\}^{\frac{1}{1+\varepsilon}}. \quad (15)$$

3. The steady state

3.1. The multiplicity of BGPs

We define a balanced growth path (hereafter BGP) where output, consumption, public debt, capital and productive public expenditures grow at a unique rate, namely γ^* ($\gamma^* = \dot{y}_t/y_t = \dot{c}_t/c_t = \dot{b}_t/b_t = \dot{k}_t/k_t = \dot{g}_t/g_t$). Since we are interested in studying the relation between tax evasion, public debt and economic growth, we express the steady-state solution by two relations of b_k as functions of γ^* .

We get a first relation between b_k^* and γ^* from equation (12)

$$b_k^* = \frac{\theta A g_k^{*\alpha}}{\gamma^*} =: \mathcal{F}(\gamma^*), \quad (16)$$

where the steady-state ratio of productive public expenditures-to-capital is a function of γ^* . We obtain its expression by combining the Keynes-Ramsey rule ($\gamma^* = \sigma^{-1}(r^* - \rho)$) and the expression of the steady-state real interest rate ($r^* = (1 - \alpha)(1 - (1 - \eta)\tau)A g_k^{*\alpha}$)

$$g_k^* = \left[\frac{\sigma \gamma^* + \rho}{A[1 - (1 - \eta)\tau](1 - \alpha)} \right]^{\frac{1}{\alpha}}. \quad (17)$$

From (15), we obtain a second relation between γ^* and b_k^*

$$b_k^* = y_k^* \left\{ \frac{[(\theta + (1 - \eta)\tau)y_k^* - g_k^*][(1 - \eta)\tau]^\varepsilon}{(1 - \alpha)[1 - (1 - \eta)\tau]y_k^{*2}} \right\}^{\frac{1}{1+\varepsilon}} =: \mathcal{G}(\gamma^*), \quad (18)$$

where $y_k^* = A g_k^{*\alpha}$.

Hence, the steady-state economic growth rate and the ratio of public debt-to-capital are obtained at the intersection of equations (16) and (18).

Proposition 1. *(Multiplicity of BGPs) For small values of $\theta \geq 0$ and $\eta \geq 0$, there exists a non-empty set of parameters, denoted \mathcal{C} , which contains two and only two*

(positive) steady-state balanced-growth paths: a low BGP (noted γ^L) and a high BGP (noted γ^H), such that $0 < \gamma^L < \gamma^H$.

Proof. We proceed in two steps. First, we can show that there are two and only two BGPs for the special case $\theta = 0$. For $\dot{b}_t = b_t = 0$ and $\gamma > 0$, the ratio of public spending-to-capital is defined by

$$g_k^* = [A(1 - \eta)\tau]^{\frac{1}{1-\alpha}} \equiv g_k^B \quad (19)$$

and since the steady-state real interest rate is such that $r^B = (1 - \alpha)[1 - (1 - \eta)\tau]A(g_k^B)^\alpha$, the associated economic growth rate is such that

$$\gamma^B = \sigma^{-1} [(1 - \alpha)[1 - (1 - \eta)\tau]A(g_k^B)^\alpha - \rho] \quad (20)$$

The couple $(\gamma^B, 0)$ denotes the “Barro solution” where $\gamma^B > 0$.

Moreover, we can obtain a no-growth solution, called the “Solow solution”, for $\theta = \gamma = 0$. In this case, equation (16) is no longer defined but we can easily extract from (17) the corresponding expression of the productive public expenditures-to-capital ratio :

$$g_k^S = \left[\frac{\rho}{A(1 - \alpha)(1 - (1 - \eta)\tau)} \right]^{\frac{1}{\alpha}} \quad (21)$$

Therefore, b_k^S is a function of the parameters of the model

$$b_k^S = y_k^S \left\{ \frac{[(1 - \eta)\tau y_k^S - g_k^S] [(1 - \eta)\tau]^\varepsilon}{(1 - \alpha)[1 - (1 - \eta)\tau] (y_k^S)^2} \right\}^{\frac{1}{1+\varepsilon}}. \quad (22)$$

Thus, the couple $(0, b_k^S)$ characterizes the second BGP of the model for $\theta = 0$.

The second step consists of generalizing the proof to the case where $\theta > 0$. In this case, it is clear that $\mathcal{F}(\gamma^*) \in C^\infty(\mathbb{R}^+)$ is a strictly decreasing and strictly convex function since $\mathcal{F}'(\gamma^*) = -\frac{\theta\rho}{(1-\alpha)[1-(1-\eta)\tau]\gamma^2} < 0$ and $\mathcal{F}''(\gamma^*) = \frac{2\theta\rho}{(1-\alpha)[1-(1-\eta)\tau]\gamma^3} > 0$. In addition, $\lim_{\gamma^* \rightarrow 0} \mathcal{F}(\gamma^*) = -\infty$ and $\lim_{\gamma^* \rightarrow +\infty} \mathcal{F}(\gamma^*) = 0$. On the other hand, we can

observe that $\mathcal{G}(\gamma^*) \in C^\infty(]0, \bar{\gamma}^*[)$ is characterized by an inverted U-shaped curve on the interval $]0, \bar{\gamma}^*[$. Indeed, from $\mathcal{G}'(\gamma^*) = 0$, we can extract a threshold of growth (noted $\hat{\gamma}^*$) which is equal to $\hat{\gamma}^* = \frac{1}{\sigma} \left[(1 - \alpha)[1 - (1 - \eta)\tau]A \left(\frac{\varepsilon\alpha[\theta + (1 - \eta)\tau]A}{1 - \alpha(1 - \varepsilon)} \right)^{\frac{\alpha}{1 - \alpha}} - \rho \right]$. Besides, it is quite obvious that $\mathcal{G}''(\gamma^*) < 0$ and that $\mathcal{G}(\gamma^*)$ is therefore concave. Finally, we have $\lim_{\gamma^* \rightarrow 0} \mathcal{G}(\gamma^*) = b_k^S$ and $\lim_{\gamma^* \rightarrow \bar{\gamma}^*} \mathcal{G}(\gamma^*) = 0$.

Since $\bar{\gamma}^* > \gamma^S$, according to the intermediate value theorem, there is a non-empty set of parameters \mathcal{C} such that $\mathcal{F}(\gamma^*)$ and $\mathcal{G}(\gamma^*)$ intersect twice in the plane $\mathbb{R}^+ \times \mathbb{R}^+$ and give rise to two real and positive solutions for the steady-state growth rate noted γ_1^* and γ_2^* . Therefore, we define the low BGP as $\gamma^L \equiv \min(\gamma_1^*, \gamma_2^*)$ and the high BGP as $\gamma^H \equiv \max(\gamma_1^*, \gamma_2^*)$. \square

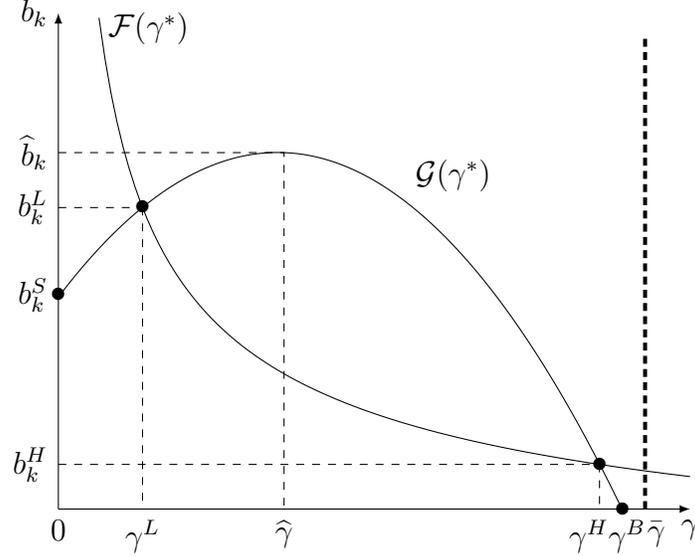


Figure 1: The steady-state solutions

In our setup, the multiplicity of BGPs comes from the interaction between the Keynes Ramsey rule and the government budget constraint. The intuitive explanation is the following. A low rate of economic growth amplifies the crowding-out effect on productive expenditures by increasing the debt burden which, in turn, further increases the public debt-to-capital ratio. Conversely, a high rate of economic growth,

by increasing the real interest rate, increases the return of the government bonds which further reduces public debt accumulation. This interaction between economic growth and public debt generates multiplicity and leads to the emergence of two BGPs. Therefore, there are two equilibria in the long run: a low-growth and high-public debt equilibrium and a high-growth and low-public debt equilibrium.

In what follows, we will successively analyze the effects of tax evasion on growth and public debt accumulation in low-growth and high-growth economies, respectively.

3.2. The low steady-state

The low BGP is characterized by a low growth solution and a positive ratio of public debt-to-capital. Contrary to the Solow solution which defines a no growth solution ($\gamma^S = \theta = 0$), the low BGP gives rise to an economic growth rate slightly higher than zero. The Solow solution is reached for $\theta = 0$ while the low BGP is reached for $\theta > 0$ and $\gamma^L \rightarrow \gamma^S$ (with $\gamma^L > \gamma^S$).

Thus, from (6) and (18), we can approximate the expression of the low steady-state by the following two relations

$$\gamma^L \approx \frac{\theta}{b_y^S}, \quad (23)$$

where

$$b_y^S = \left\{ \frac{[(1-\eta)\tau - A^{-1}(g_k^S)^{1-\alpha}] [(1-\eta)\tau]^\varepsilon}{(1-\alpha)(1-(1-\eta)\tau)y_k^S} \right\}^{\frac{1}{1+\varepsilon}}. \quad (24)$$

Proposition 2 establishes the steady-state impact of tax evasion on economic growth in the neighborhood of the low BGP.

Proposition 2. *(Effects of tax evasion on growth and public debt in the neighborhood of the low BGP) In the neighborhood of the low BGP,*

- (i) *there is a negative relation between tax evasion and public debt accumulation.*

(ii) *tax evasion leads to an increase in economic growth.*

Proof. See Appendix B

In the neighborhood of the low BGP, any increase in tax evasion requires a lower debt and a higher growth to be compatible with the steady-state. The intuitive explanation of this result is the following. Tax evasion reduces tax revenues. Other things being equal, this leads to an increase in the risk premium, the return of the government bonds and consequently the debt burden in the long run. Thus, in the absence of another instrument of public finance (such as seigniorage for instance), the government has no choice but to decrease the deficit-to-output ratio to be able to pay down the debt. This reduction in public debt reduces the unproductive public expenditures related to the debt burden and allows the government to have more resources to finance productive public spending. Finally, since public debt and growth are negatively related in the long run, the negative impact of tax on public debt implies a positive impact on economic growth.

3.3. *The high steady-state*

Proposition 3. *(Effects of tax evasion on growth and public debt in the neighborhood of the high BGP) In the neighborhood of the high BGP,*

(i) *the relation between tax evasion and public debt accumulation is characterized by a U-shaped curve.*

(ii) *there is an inverted U-shaped relation between tax evasion and long-run growth.*

Proof. See Appendix C

In the neighborhood of the high BGP, we find a critical level of tax evasion ($\hat{\eta} \in (0, 1)$) such that the relation between tax evasion and growth is reversed. At low levels, the impact of tax evasion on growth is positive. At high levels, the impact of

tax evasion on growth is negative. This is explained by the dual effect that tax evasion exerts on long-run growth. On the one hand, tax evasion positively affects long-run growth by stimulating capital accumulation and investment through the channel of the disposable income of households. On the other hand, tax evasion generates a reduction of the productive public expenditures provided by the government which has a detrimental impact on long-run growth. When the first effect dominates, the impact of tax evasion on growth is positive and conversely. As before, since public debt and growth are negatively related, the effects of tax evasion on growth and public debt are symmetric. Therefore, the threshold of tax evasion that maximizes growth minimizes the public debt ratio. At low levels, tax evasion reduces public debt (as in the neighborhood of the low BGP) but high levels of tax evasion lead to an increase in public debt in the long run.

4. Local dynamics

4.1. The reduced-form

To study the dynamics of the model outside the steady-state, we compute a two-variables reduced form (see Appendix D). Relation (25) describes the Keynes-Ramsey rule that characterizes the consumption behavior while relation (26) establishes the law of motion of productive public expenditures.

$$\frac{\dot{c}_k}{c_k} = \frac{r_t - \rho}{\sigma} + c_k + g_k - y_k \quad (25)$$

$$\frac{\dot{g}_k}{g_k} = \frac{\mathcal{M}(g_k)}{g_k} \left[\frac{\theta A g_k^\alpha}{b_k} + c_k + g_k - y_k \right] \quad (26)$$

where

$$\mathcal{M}(g_k) = \frac{(1 + \varepsilon)[(\theta + (1 - \eta)\tau)y_k - g_k]g_k}{\varepsilon\alpha[(\theta + (1 - \eta)\tau)y_k - g_k] - (1 - \alpha)g_k} \quad (27)$$

In the reduced-form composed of (25)-(26), there is one jump variable (the ratio of consumption-to-capital c_k) and one predetermined variable (the ratio of productive public expenditures-to-capital g_k). The ratio of productive public expenditures-to-capital is a predetermined variable since g_k depends on b_k and b_k can never jump because b_t and k_t are predetermined $\forall t$.

To study the local stability of the BGPs, we resort to a linearized form of the system (25)-(26) in the neighborhood of the steady-state i ($i \in \{L, H\}$).

$$\begin{pmatrix} \dot{c}_k \\ \dot{g}_k \end{pmatrix} = \mathbf{J}^i \begin{pmatrix} c_k - c_k^{*i} \\ g_k - g_k^{*i} \end{pmatrix} \quad i \in \{L, H\} \quad (28)$$

where \mathbf{J}^i is the Jacobian matrix in the neighborhood of BGP i . Since there is one predetermined variable and one jump variable, the Blanchard-Kahn conditions are fulfilled if and only if the Jacobian matrix contains two opposite-signs eigenvalues.

Thus, to analyze the local determinacy or indeterminacy of the BGPs, we study the following characteristic polynomial of degree 2 associated with the steady-state i :

$$\mathcal{P}^i(\lambda) = \lambda^2 - \mathcal{T}(\mathbf{J}^i)\lambda + \mathcal{D}(\mathbf{J}^i) = 0 \quad (29)$$

where the roots of the characteristic polynomial (λ_1 and λ_2) correspond to the eigenvalues of the Jacobian matrix and $\mathcal{T}(\mathbf{J}^i)$ and $\mathcal{D}(\mathbf{J}^i)$ are respectively the trace and the determinant of the Jacobian matrix in the neighborhood of the steady-state i . In order to study the topological behavior of each steady-state, the following proposition successively establishes the determinant and the trace of the high and the low BGP for low values of the deficit target⁵.

Proposition 4. *(Determinant and trace of the Jacobian matrix in the neighborhood of BGP i). For low values of θ (i.e. $\theta \rightarrow 0$), the determinant and the trace of the Jacobian matrix in the neighborhood of BGP i are the following.*

⁵expliquer

(i) In the neighborhood of the Barro BGP :

$$\mathcal{D}(\mathbf{J}^B) = -\gamma^B c_k^B$$

$$\mathcal{T}(\mathbf{J}^B) = -\gamma^B + c_k^B$$

(ii) In the neighborhood of the Solow BGP :

$$\mathcal{D}(\mathbf{J}^S) = -\tilde{\mathcal{M}}(g_k^S) c_k^S (1 - \alpha) [1 - (1 - \eta)\tau] \sigma^{-1} A(g_k^S)^{\alpha-1}$$

$$\mathcal{T}(\mathbf{J}^S) = c_k^S + \tilde{\mathcal{M}}(g_k^S) \left[1 - \alpha A(g_k^S)^{\alpha-1} \right]$$

where $\tilde{\mathcal{M}}(g_k^S) = \mathcal{M}(g_k^S)|_{\theta \rightarrow 0}$.

PROOF. See Appendix E.

4.2. Local stability of the Barro BGP

From Proposition 3, we can directly establish the determinacy of the Barro BGP as follows.

Proposition 5. *(Determinacy of the Barro BGP) The Barro BGP is well-determined (i.e. saddle-path stable) for any values of the parameters.*

Proof. From Proposition 3, it is clear that the determinant of the Barro BGP is always negative. Therefore, $\mathcal{P}^B(0) = -\gamma^B c_k^B < 0$ and the Jacobian matrix in the neighborhood of the Barro BGP contains two opposite-signs real eigenvalues. This ensures the fulfillment of the Blanchard-Kahn conditions and, therefore, the local determinacy of the Barro BGP. \square

4.3. Local stability of the Solow BGP

The determinacy of the Solow BGP is more complicated to establish since it crucially depends on the sign of $\tilde{\mathcal{M}}(g_k^S)$ and the value of the tax evasion parameter. Specifically, three configurations are possible. First, if $\mathcal{D}(\mathbf{J}^S) < 0$, the Jacobian matrix would possess two opposite-signs eigenvalues and the Solow BGP would be saddle-path stable. Second, if $\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) > 0$, the Jacobian matrix would contain two eigenvalues with positive real parts and the BGP would be overdetermined (unstable). This configuration, which corresponds for instance to the case where $\varepsilon = 0$, has extensively been studied by Minea and Villieu (2012). Finally, if $\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) < 0$, the two eigenvalues of the Jacobian matrix would have negative real parts and the Solow BGP would be undetermined (stable). The transition between the second case and the third case is featured by a Hopf bifurcation. At that point, the Jacobian matrix contains a pair of complex conjugate eigenvalues (with real parts equal to zero).

In what follows, in order to characterize the local dynamics of the Solow BGP, we will consider the parameter describing the sensitivity of the risk premium (ε) as our bifurcation parameter. This parameter that generates the complexity of the dynamics of the Solow BGP. In a model close to ours but without risk premium and tax evasion ($\varepsilon = \eta = 0$), Minea and Villieu (2012) have indeed shown that the Solow BGP is always unstable. Therefore, we will analyze the values of ε at which $\tilde{\mathcal{M}}(g_k^S)$, and then the determinant and the trace of the Jacobian matrix, change sign in the neighborhood of the Solow BGP.

Lemma 1. *The value of ε at which $\tilde{\mathcal{M}}(g_k^S)$ changes sign (noted $\tilde{\varepsilon}$) can be expressed as*

$$\tilde{\varepsilon} = \frac{(1 - \alpha)g_k}{\alpha(1 - \eta)\tau y_k - \alpha g_k}$$

Proposition 6. (*Determinacy of the Solow BGP*) *The topological properties of the Solow BGP can be summarized as follows :*

- (i) *if $\varepsilon > \tilde{\varepsilon}$, the Solow BGP is saddle-path stable.*
- (ii) *if $\varepsilon < \tilde{\varepsilon}$ and $\varepsilon = \varepsilon^h$ (where $\varepsilon^h < \tilde{\varepsilon}$), a Hopf bifurcation occurs in the neighborhood of the Solow BGP. Thus, when $\varepsilon < \varepsilon^h$, the Solow BGP is overdetermined (unstable) and when $\varepsilon > \varepsilon^h$, the Solow BGP becomes undetermined (stable).*

Proof. Let us successively consider the case $\varepsilon > \tilde{\varepsilon}$ and the case $\varepsilon < \tilde{\varepsilon}$.

- (i) If $\varepsilon > \tilde{\varepsilon}$, then $\tilde{\mathcal{M}}(g_k^S) > 0$ and $\mathcal{D}(\mathbf{J}^S) < 0$. In that case, the Jacobian matrix in the neighborhood of the Solow BGP contains two opposite signs eigenvalues. In this case, the Solow BGP is therefore saddle-path stable.
- (ii) If $\varepsilon < \tilde{\varepsilon}$, then $\tilde{\mathcal{M}}(g_k^S) < 0$ and three different cases are possible. First, if $\varepsilon = \varepsilon^h$ where $\varepsilon^h < \tilde{\varepsilon}$, we obtain $\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) = 0$. In this case, the Jacobian matrix contains two complex conjugate eigenvalues and a Hopf bifurcation occurs. Second, if $\varepsilon^h < \varepsilon < \tilde{\varepsilon}$, then the determinant of the Jacobian matrix in the neighborhood of the Solow BGP is positive while its trace is negative ($\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) < 0$). In this case, the Solow BGP is stable. Third, if $\varepsilon < \varepsilon^h < \tilde{\varepsilon}$, then both the determinant and the trace of the Jacobian matrix in the neighborhood of the Solow BGP are positive ($\mathcal{D}(\mathbf{J}^S) > 0$ and $\mathcal{T}(\mathbf{J}^S) > 0$). Therefore, the Solow BGP is overdetermined (unstable).

□

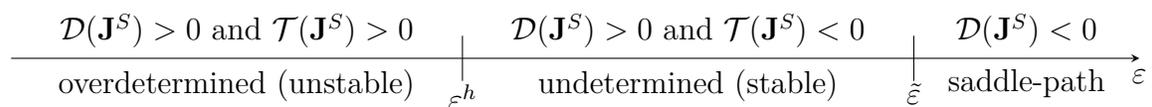


Figure 2: Local stability of the Solow BGP depending on the parameter ε

Lemma 2. *The Hopf bifurcation occurs at the unique point ε^h defined as*

$$\varepsilon^h = \frac{(1 - \alpha)g_k^S c_k^S + (\alpha y_k^S - g_k^S)[(1 - \eta)\tau y_k^S - g_k^S]}{[(1 - \eta)\tau y_k^S - g_k^S][g_k^S - \alpha(y_k^S - c_k^S)]}$$

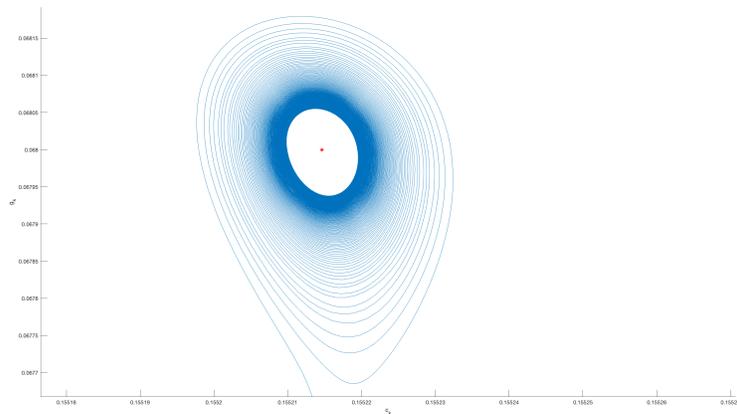


Figure 3: The Hopf bifurcation

To illustrate our results, Figure 3 depicts the Hopf bifurcation (red point) that occurs at the Solow BGP. Our baseline calibration is based on reasonable values of parameters. The discount rate is fixed at $\rho = 0.1$ and the risk-aversion coefficient is $\sigma = 1$. Regarding the technology, the total factor productivity parameter is set at $A = 0.5$ and the share of productive public spending is $\alpha = 0.3$ in order to obtain a capital share close to that (0.715) of Gomme et al. (2011). Following Trabandt and Uhlig (2011) and Gomes et al. (2013), the income tax rate is fixed at $\tau = 0.4$. In order to obtain a realistic economic growth rate, we set $\eta = 0.1$. Initially, we consider $\theta = 0$ to illustrate the zero-growth case. Thereafter, we will analyze the case where $\theta > 0$ to numerically explore the existence of Hopf bifurcations in the neighborhood of the low BGP (and not only in the neighborhood of the Solow BGP). Table ?? provides several numerical simulations showing the robustness of the Hopf bifurcation's occurrence in the neighborhood of the Solow and the low BGP for numerous sets of parameters.

5. The model with endogenous tax evasion

In this section, we extend the previous framework to the case where tax evasion is endogenous. To this end, we now consider that households make an effort e_t to evade taxes such that $e_t \in (1, \infty)$ and the government invests resources noted h_t to fight against tax evasion. Both e_t and h_t are endogenous variables. Thus, households no longer evade $\eta\tau y_t$ but a portion of taxes $\eta(e_t)\tau y_t$ where

$$\eta(e_t) = \eta_0 \Gamma(e_t), \quad (30)$$

where η_0 corresponds to the initial level of tax evasion and $\Gamma(e_t)$ is an increasing and concave function ($\Gamma'(e_t) > 0$ and $\Gamma''(e_t) < 0$). In addition, $\eta(1) = \eta_0$ and $\lim_{e_t \rightarrow \infty} \eta(e_t) = 1/\eta_0$ so that $\eta(e_t) \in (\eta_0, 1)$. To simply satisfy these properties, we assume that $\Gamma(e_t) := e_t^\beta$ where $\beta \in (0, 1)$.

Thus, the new household budget constraint is expressed as follows

$$\dot{k}_t + \dot{b}_t = [1 - \mathcal{P}(\cdot)] R_t b_t + [1 - (1 - \eta(e_t))\tau] y_t - c_t - h_t e_t, \quad (31)$$

where h_t denotes the c

From the first order condition of the hamiltonian with respect to e_t , we obtain the level of efforts chosen by households e_t to evade taxes

$$e_t = \left(\frac{\eta_0 \beta \tau y_k}{h_k} \right)^{\frac{1}{1-\beta}}. \quad (32)$$

From (32), we can notably observe that contractionary fiscal policies lead to more tax evasion, in line with Allingham and Sandmo (1972) and Cerqueti and Coppier (2011).

Henceforth, the government faces the following budget constraint

$$\dot{b}_t = R_t b_t + g_t - [1 - \eta(e_t)]\tau y_t - h_t e_t, \quad (33)$$

where the expression of R_t is given by (14) and the real interest rate is similar to that of the model with exogenous tax evasion: $r_t = (1 - \alpha)[1 - (1 - \eta(e_t))\tau]y_k$.

Finally, the amount of resources invested by the government to combat tax evasion h_t evolves according to the following dynamics

$$\dot{h}_t = \xi [\eta(e_t) - \bar{\eta}] h_t, \quad (34)$$

where $\bar{\eta}$ corresponds to the level of tax evasion targeted by the government and ξ is a strictly positive parameter.

5.1. The long-run solution

As in the exogenous tax evasion case, the steady-state solution is obtained at the intersection between two relations derived from the the Keynes-Ramsey rule, the deficit rule and the government budget constraint. However, when tax evasion is endogenous, it is rather difficult to extract analytical expressions for the long run economic growth rate and the long run public debt-to-capital ratio, even in the case where $\theta = 0$. Consequently, we resort to numerical simulation (see figure 4) and show that the steady-state solution is similar to that of the exogenous tax evasion case. As previously, when tax evasion is endogenous, the model is characterized by a low-growth and high-public debt solution and a high-growth and low-public debt solution.

We now want to study the long run effects of a change in the level of tax evasion targeted by the government and the elasticity of the risk premium. Table 1 and Table 2 respectively summarize the impact of an increase in the tax evasion target and the elasticity of the risk premium on public debt and growth in the neighborhood of both BGPs.

Along the high BGP, we can observe that any increase in the targeted level of tax evasion allows increasing the rate of economic growth and reducing the ratio of public debt-to-capital. In contrast, along the low BGP, an increase in the tax evasion target leads to a decrease in growth and an increase in public debt. The intuitive

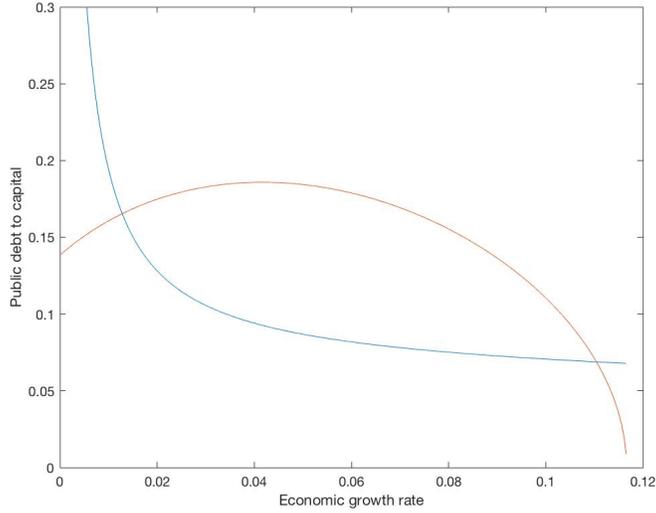


Figure 4: The steady-state solution

$\bar{\eta}$	γ^L	b_k^L	γ^H	b_k^H
0	0.0117	0.1657	0.1086	0.0658
0.1	0.0128	0.1566	0.1104	0.0655
0.2	0.0142	0.1472	0.1113	0.0654
0.3	0.0159	0.1307	0.1111	0.0624
0.4	0.0182	0.1155	0.1094	0.0599
0.5	0.0215	0.1011	0.1055	0.0579
0.6	0.0266	0.0873	0.0978	0.0566
0.7	0.0374	0.0728	0.0806	0.0571

Table 1: Effects of a change in $\bar{\eta}$ on long run growth and public debt

explanation of this nonlinear effect is the following. In high-growth economies, a higher target of tax evasion allows (i) improving the productivity of public expenditures and consequently enhancing growth and (ii) reducing the debt burden, which, in turn, decreases the stock of debt. In low-growth economies, tax evasion increases the charge burden by increasing the risk premium since $\eta(e^*) \approx (\bar{\eta})^{\frac{1}{\beta}}$, leading to higher public debt and a lower growth in the long run.

Table 2 shows that an increase in the elasticity of the risk premium has similar

ε	γ^L	b_k^L	γ^H	b_k^H
0	0.0014	0.9923	0.1088	0.0690
1	0.0128	0.1653	0.1104	0.0688
2	0.0271	0.1105	0.1118	0.0686
3	0.0383	0.0956	0.1129	0.0684
4	0.0464	0.0891	0.1138	0.0682
5	0.0523	0.0855	0.1144	0.0681
6	0.0567	0.0833	0.1150	0.0681
7	0.0601	0.0818	0.1153	0.0680
8	0.0627	0.0807	0.1156	0.0679
9	0.0648	0.0799	0.1159	0.0679
10	0.0665	0.0793	0.1160	0.0679
20	0.0743	0.0767	0.1166	0.0678

Table 2: Effects of a change in ε on long run growth and public debt

effects in terms of growth and public debt in the neighborhood of both BGP. It increases the rate of growth and decrease the stock of debt. This result can be explained by the fact that a higher elasticity of the premium decreases the cost of public debt for the government and consequently mitigates the crowding-out effect on productive public expenditures, which boosts economic growth. However, we can notice that the effects of a change in the elasticity of the risk premium are larger in low-growth economies than in high growth economies because.

5.2. Transitional dynamics

The model with endogenous tax evasion can be summarized by a three-variable reduced form in c_k , b_k and h_k

$$\begin{cases} \dot{c}_k = \left(\frac{r_t - \rho}{\sigma} + c_k + g_k - y_k \right) c_k \\ \dot{h}_k = [\xi (\eta(e_t) - \bar{\eta}) + c_k + g_k - y_k] h_k \\ \dot{g}_k = \mathcal{K}(\cdot) \left\{ (1 + \varepsilon) \left[\frac{\theta y_k}{b_k} + c_k + g_k - y_k \right] - x(\cdot) \frac{\dot{h}_k}{h_k} \right\} \end{cases} \quad (35)$$

where the expression of the constant $\mathcal{K}(\cdot)$ and the function $x(\cdot) \equiv x(g_k, h_k)$ are given in Appendix G.

6. Conclusion

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Appendix A: Resolution of the model

The household's program is solved by maximizing (1) subject to the constraints (2), (3) and (4). The initial values k_0 and b_0 are given. The current hamiltonian associated with the household's maximization program is

$$\mathcal{H}_c = u(c_t) + \lambda_{1t} \{ [1 - \mathcal{P}(\cdot)] R_t b_t + y_t^d - c_t - z_t \} + \lambda_{2t} z_t \quad (\text{A.1})$$

where z_t is a slack variable defined as $z_t = \dot{k}_t$ and λ_{1t} and λ_{2t} are the co-state variables respectively associated with b_t and k_t .

The first order conditions of the household's maximization program with respect to c_t , z_t , b_t and k_t are

$$/c_t \quad \lambda_{1t} = u'(c_t) = c_t^{-\sigma} \quad (\text{A.2})$$

$$/z_t \quad \lambda_{1t} = \lambda_{2t} \quad (\text{A.3})$$

$$/b_t \quad \frac{\dot{\lambda}_{1t}}{\lambda_{1t}} = \rho - [1 - \mathcal{P}(\cdot)] R_t \quad (\text{A.4})$$

$$/k_t \quad \frac{\dot{\lambda}_{2t}}{\lambda_{2t}} = \rho - (1 - \alpha) [1 - (1 - \eta)\tau] \frac{\lambda_{1t}}{\lambda_{2t}} A g_k^\alpha \quad (\text{A.5})$$

The transversality conditions are standard

Appendix B: Proof of proposition 2

Let us replace y_k^S by its expression in (24) and rewrite the steady-state public debt-to-GDP ratio in the neighborhood of the Solow BGP as follows :

$$b_y^S = \left\{ \frac{f_1^S(\eta) [f_2^S(\eta) - A^{-1}f_3^S(\eta)]}{\rho} \right\}^{\frac{1}{1+\varepsilon}} \equiv \mathcal{B}(\eta) \quad (\text{B.1})$$

where :

$$f_1^S(\eta) = [(1 - \eta)\tau]^\varepsilon \quad (\text{B.2})$$

$$f_2^S(\eta) = (1 - \eta)\tau \quad (\text{B.3})$$

$$f_3^S(\eta) = (g_k^S)^{1-\alpha} = \left[\frac{\rho}{A(1 - \alpha)(1 - (1 - \eta)\tau)} \right]^{\frac{1-\alpha}{\alpha}} \quad (\text{B.4})$$

Then, we can determine the first derivatives of equations (B.2)-(B.3) with respect to η :

$$f_1'^S(\eta) = -\varepsilon\tau [(1 - \eta)\tau]^{\varepsilon-1} < 0 \quad (\text{B.5})$$

$$f_2'^S(\eta) = -\tau < 0 \quad (\text{B.6})$$

$$f_3'^S(\eta) = -\frac{A(1 - \alpha)^2\tau g_k^S}{\alpha\rho} < 0 \quad (\text{B.7})$$

Hence, we can show that the impact of tax evasion on the ratio of public debt-to-

GDP is negative in the neighborhood of the Solow BGP :

$$\mathcal{B}'(\eta) = \frac{(b_y^S)^{-1} f_1'^S(\eta) [f_2^S(\eta) - A^{-1} f_3^S(\eta)] + f_1^S(\eta) [f_2'^S(\eta) - A^{-1} f_3'^S(\eta)]}{1 + \varepsilon \rho} < 0 \quad (\text{B.8})$$

since $f_2^S(\eta) - A^{-1} f_3^S(\eta) < 0$ for positive values of the tax rate.

Appendix C: Proof of proposition 3

To study the impact of tax evasion on growth in the neighborhood of the high BGP, we determine the first order condition of (20) with respect to the parameter of tax evasion and extract a critical level of tax evasion (noted $\hat{\eta}$) such that its long-run effect on growth is reversed :

$$\frac{\partial \gamma^B}{\partial \eta} \geq 0 \quad \text{if } \eta \leq \hat{\eta} \quad (\text{C.1})$$

where

$$\hat{\eta} = 1 - \frac{\alpha}{\tau} \quad (\text{C.2})$$

Contrary to the results found by Barro (1990) and Futagami *et al.* (1993), we assume the tax rate fixed by the government (τ) to be higher than the elasticity of productive public expenditures (α) because of the presence of tax evasion in the economy. Therefore, $0 < \hat{\eta} < 1$.

In addition, the second order condition shows that the function γ^B is always strictly concave in η . This ensures that the threshold $\hat{\eta}$ is a maximum.

$$\frac{\partial^2 \gamma^B}{\partial \eta^2} = - \frac{\alpha [A(1 - \eta)\tau]^{1-\alpha} [1 + (1 - \eta)\tau - 2\alpha]}{\sigma(1 - \alpha)(\eta - 1)^2} < 0 \quad (\text{C.3})$$

Appendix D: The reduced form of the model

By defining intensive variables $c_k = c_t/k_t$, $g_k = g_t/k_t$, $b_k = b_t/k_t$ and $y_k = y_t/k_t$, we obtain the reduced form of the model. From the Keynes-Ramsey rule, we get

$$\frac{\dot{c}_k}{c_k} = \frac{r_t - \rho}{\sigma} - \gamma_k \quad (\text{D.1})$$

where $\gamma_k \equiv \frac{\dot{k}_t}{k_t}$ is obtained using the IS equilibrium:

$$\gamma_k = y_k - g_k - c_k \quad (\text{D.2})$$

The local stability of the model can be studied from a reduced-form in c_k and g_k or in c_k and b_k . We successively present both alternatives. For simplicity, we will resort to the reduced-form in c_k and g_k to study the local stability of the low BGP and to the reduced-form in c_k and b_k for the local stability of the high BGP.

Appendix D.1: The reduced form in c_k and g_k

The government budget constraint provides the following relation

$$\frac{\dot{b}_k}{b_k} = \frac{1}{1 + \varepsilon} \left[\varepsilon \alpha - \frac{(1 - \alpha)g_k}{(\theta + (1 - \eta)\tau)y_k - g_k} \right] \frac{\dot{g}_k}{g_k} \quad (\text{D.3})$$

In addition, from (8), we also get

$$\frac{\dot{b}_k}{b_k} = \frac{\theta y_k}{b_k} - \gamma_k \quad (\text{D.4})$$

Hence, we can easily extract the dynamics of productives public expenditures over time by combining (D.3) and (D.4)

$$\frac{\dot{g}_k}{g_k} = \frac{(1 + \varepsilon) [(\theta + (1 - \eta)\tau)y_k - g_k] \left[\frac{\theta y_k}{b_k} - \gamma_k \right]}{\varepsilon \alpha [(\theta + (1 - \eta)\tau)y_k - g_k] - (1 - \alpha)g_k} \quad (\text{D.5})$$

Thus, replacing γ_k by its expression, we obtain the following reduced-form in c_k and g_k :

$$\begin{cases} \dot{c}_k = \left(\frac{r_t - \rho}{\sigma} + c_k + g_k - y_k \right) c_k \\ \dot{g}_k = \mathcal{M}(g_k) \left[\frac{\theta A g_k^\alpha}{b_k} + c_k + g_k - y_k \right] \end{cases} \quad (\text{D.6})$$

where

$$\mathcal{M}(g_k) = \frac{(1 + \varepsilon)[(\theta + (1 - \eta)\tau)y_k - g_k]g_k}{\varepsilon \alpha [(\theta + (1 - \eta)\tau)y_k - g_k] - (1 - \alpha)g_k} \quad (\text{D.7})$$

Appendix D.2: The reduced form in c_k and b_k

We obtain the reduced-form in c_k and b_k from equations (8) and (9). After some simple manipulations, we get :

$$\begin{cases} \dot{c}_k = \left(\frac{r_t - \rho}{\sigma} + c_k + g_k - y_k \right) c_k \\ \dot{b}_k = \theta y_k + (c_k + g_k - y_k) b_k \end{cases} \quad (\text{D.8})$$

where g_k is an implicit function of b_k .

Appendix E: Local stability of the BGPs

In the neighborhood the low BGP (for $\theta \rightarrow 0$), the Jacobian matrix associated with the system (D.6) is :

$$\mathbf{J}^S = \begin{bmatrix} c_k^S & \left[1 - \alpha A (g_k^S)^{\alpha-1} (1 - (1 - \alpha)(1 - (1 - \eta)\tau)\sigma^{-1})\right] c_k^S \\ \tilde{\mathcal{M}}(g_k^S) & \tilde{\mathcal{M}}(g_k^S) \left[1 - \alpha A (g_k^S)^{\alpha-1}\right] \end{bmatrix} \quad (\text{E.1})$$

From (E.1), we easily obtain $\mathcal{D}(\mathbf{J}^S) = -\tilde{\mathcal{M}}(g_k^S) c_k^S (1 - \alpha) \alpha (1 - (1 - \eta)\tau) \sigma^{-1} A (g_k^S)^{\alpha-1}$ and $\mathcal{T}(\mathbf{J}^S) = c_k^S + \tilde{\mathcal{M}}(g_k^S) \left[1 - \alpha A (g_k^S)^{\alpha-1}\right]$ where $\tilde{\mathcal{M}}(g_k^S)$ corresponds to $\mathcal{M}(g_k)$ in the neighborhood of the low steady-state for $\theta \rightarrow 0$:

$$\tilde{\mathcal{M}}(g_k^S) = \frac{(1 + \varepsilon)[(1 - \eta)\tau y_k^S - g_k^S] g_k^S}{\varepsilon \alpha [(1 - \eta)\tau y_k^S - g_k^S] - (1 - \alpha) g_k^S} \quad (\text{E.2})$$

Notice that since the growth rate in the neighborhood of the low BGP tends towards 0 ($\gamma^S = 0$), the consumption-to-capital ratio is expressed as : $c_k^S = y_k^S - g_k^S$.

In the neighborhood the high BGP (for $\theta \rightarrow 0$), the Jacobian matrix associated with the system (D.8) is :

$$\mathbf{J}^B = \begin{bmatrix} c_k^B \\ 0 & -\gamma^B \end{bmatrix} \quad (\text{E.3})$$

Hence, it is clear that the determinant of (E.3) is always strictly negative : $\mathcal{D}(\mathbf{J}^B) = -\gamma^B c_k^B < 0$.

Appendix F: Resolution of the model with endogenous tax evasion and steady-state solutions

In the endogenous tax evasion case, the steady-state solution of the model can be obtained at the intersection of the following two relations between long run growth and the public debt-to-capital ratio. The first relation comes from the definition of the public debt-to-GDP target while the second relation comes from the government

budget constraint

$$b_k^1(\gamma) = \frac{\theta y_k}{\gamma}, \quad (\text{F.1})$$

$$b_k^2(\gamma) = y_k \left\{ \frac{[\theta + (1 - (1 - \beta)\eta(e))\tau]y_k - g_k}{[(1 - \alpha)(1 - (1 - \eta(e))\tau)] y_k^2 [(1 - \eta(e))\tau]^{-\varepsilon}} \right\}^{\frac{1}{1+\varepsilon}}, \quad (\text{F.2})$$

where

$$g_k = \left\{ \frac{(\gamma\sigma + \rho)A}{(1 - \alpha)[1 - (1 - \eta(e))\tau]} \right\}^{\frac{1}{\alpha}}, \quad (\text{F.3})$$

and

$$\eta(e) = \left(\frac{\gamma}{\xi} + \bar{\eta} \right)^{\frac{1}{\beta}} \quad (\text{F.4})$$

Appendix G: The model with endogenous tax evasion: reduced form and topological behavior of the BGPs

Outside the steady state, the model with endogenous tax evasion gives rise to a three-variable reduced form in c_k , h_k and h_k

$$\begin{cases} \dot{c}_k = \left(\frac{r_t - \rho}{\sigma} + c_k + g_k - y_k \right) c_k \\ \dot{h}_k = [\xi (\eta(e_t) - \bar{\eta}) + c_k + g_k - y_k] h_k \\ \dot{g}_k = \mathcal{K}(\cdot) \left\{ (1 + \varepsilon) \left[\frac{\theta y_k}{b_k} + c_k + g_k - y_k \right] - x(\cdot) \frac{\dot{h}_k}{h_k} \right\} \end{cases} \quad (\text{G.1})$$

where $r_t = (1 - \alpha)[1 - (1 - \eta(e_t))\tau]Ag_k^\alpha$ with $e_t = \eta_0 e_t^\beta$ and the expression of e_t is

given by

$$e_t = \left(\frac{\eta_0 \beta \tau y_k}{h_k} \right)^{\frac{1}{1-\beta}}. \quad (\text{G.2})$$

In addition

$$\mathcal{K}(\cdot) = \frac{[(\theta + (1 - \eta(e_t))\tau)y_k - g_k + e_t h_k]g_k}{[(\theta + (1 - \eta(e_t))\tau)y_k - g_k + e_t h_k] [\varepsilon\alpha - \alpha(\Psi^1(\cdot) + \Psi^2(\cdot))] - (1 - \alpha)g_k + e_t h_k}, \quad (\text{G.3})$$

$$x(\cdot) = \Psi^1(\cdot) + \Psi^2(\cdot) + \frac{e_t h_k}{(\theta + (1 - \eta(e_t))\tau)y_k - g_k + e_t h_k}, \quad (\text{G.4})$$

where

$$\Psi^1(\cdot) = \frac{\beta \tau \eta(e_t)}{(1 - \beta)(1 - (1 - \eta(e_t))\tau)} \quad \text{and} \quad \Psi^2(\cdot) = \frac{\beta \varepsilon \eta(e_t)}{(1 - \beta)(1 - \eta(e_t))}$$

Finally, the government budget constraint provides the new expression of the public-debt to capital ratio

$$b_k = y_k \left\{ \frac{[\theta + (1 - (1 - \beta)\eta(e_t))\tau]y_k - g_k}{[(1 - \alpha)(1 - (1 - \eta(e_t))\tau)] y_k^2 [(1 - \eta(e_t))\tau]^{-\varepsilon}} \right\}^{\frac{1}{1+\varepsilon}}. \quad (\text{G.5})$$

In order to examine the local stability of the BGPs, we linearize the system G.1 in the neighborhood of the steady-state i (where $i \in (L, H)$)

$$\begin{pmatrix} \dot{c}_k \\ \dot{h}_k \\ \dot{g}_k \end{pmatrix} = \mathbf{J}^i \begin{pmatrix} c_k - c_k^i \\ h_k - h_k^i \\ g_k - g_k^i \end{pmatrix}, \quad (\text{G.6})$$

Let us first define the derivatives of $\eta(e_t)$, r_t , $\Psi^1(\cdot)$, $\Psi^2(\cdot)$ and $x(\cdot)$ with respect to

g_k and h_k

$$\begin{aligned} \eta_g^i &:= \left. \frac{\partial \eta(e_t)}{\partial g_k} \right|_{*i} = \frac{\alpha \beta}{(1-\beta)} \frac{\eta(e^i)}{g_k^i}, \quad \eta_h^i := \left. \frac{\partial \eta(e_t)}{\partial h_k} \right|_{*i} = -\frac{\beta}{(1-\beta)} \frac{\eta(e^i)}{h_k^i}, \quad r_h^i := \left. \frac{\partial r_t}{\partial h_k} \right|_{*i} = -\left(\frac{1-\alpha}{1-\beta} \right) e^i, \\ r_g^i &:= \left. \frac{\partial r_t}{\partial g_k} \right|_{*i} = \alpha(1-\alpha) \left[\left(\frac{1-\alpha}{1-\beta} \right) \frac{h_k^i}{g_k^i} e^i + \left(1 - (1-\eta(e^i)) \tau \right) A(g_k^i)^{\alpha-1} \right], \\ \tilde{\Psi}_g^1(\cdot) &:= \left. \frac{\partial \Psi^1(\cdot)}{\partial g_k} \right|_{*i} = \frac{\beta \tau}{1-\beta} \frac{(1-\tau) \eta_g^i}{(1-(1-\eta(e^i))\tau)^2}, \quad \tilde{\Psi}_g^2(\cdot) := \left. \frac{\partial \Psi^2(\cdot)}{\partial g_k} \right|_{*i} = \frac{\beta \varepsilon \eta_g^i}{(1-\beta)(1-\eta(e^i))^2}, \\ \tilde{\Psi}_h^1(\cdot) &:= \left. \frac{\partial \Psi^1(\cdot)}{\partial h_k} \right|_{*i} = \frac{\beta \tau}{1-\beta} \frac{(1-\tau) \eta_h^i}{(1-(1-\eta(e^i))\tau)^2}, \quad \tilde{\Psi}_h^2(\cdot) := \left. \frac{\partial \Psi^2(\cdot)}{\partial h_k} \right|_{*i} = \frac{\beta \varepsilon \eta_h^i}{(1-\beta)(1-\eta(e^i))^2}, \\ \tilde{x}_g(\cdot) &:= \left. \frac{\partial x(\cdot)}{\partial g_k} \right|_{*i} = \tilde{\Psi}_g^1(\cdot) + \tilde{\Psi}_g^2(\cdot) + \frac{e^i}{1-\beta} \frac{h_k^i \alpha [1 - (1-\beta)\eta(e^i)] \tau y_k^i + (1-\alpha-\beta) g_k^i}{(1-\eta(e^i)) \tau y_k^i - g_k + e^i h_k^i}, \\ \tilde{x}_h(\cdot) &:= \left. \frac{\partial x(\cdot)}{\partial h_k} \right|_{*i} = \tilde{\Psi}_h^1(\cdot) + \tilde{\Psi}_h^2(\cdot) + \frac{\beta (g_k^i - \tau y_k^i) e^i}{(1-\beta) [(1-\eta(e^i)) \tau y_k^i - g_k^i + e^i h_k^i]}. \end{aligned}$$

The Jacobian matrix of the system G.1 in the neighborhood of the steady-state i can be written as

$$\mathbf{J}^i = \begin{bmatrix} c_k^i & \sigma^{-1} r_h^i c_k^i & C_g^i \\ h_k^i & \xi \eta_h^i h_k^i & H_g^i \\ G_c^i & G_h^i & G_g^i \end{bmatrix} \quad (\text{G.7})$$

where

$$\begin{aligned} C_g^i &:= \left. \frac{\partial \dot{c}_k}{\partial g_k} \right|_{*i} = \left[1 + \sigma^{-1} r_g^i - \alpha A(g_k^i)^{\alpha-1} \right] c_k^i \\ H_g^i &:= \left. \frac{\partial \dot{h}_k}{\partial g_k} \right|_{*i} = \left[1 + \xi \eta_g^i - \alpha A(g_k^i)^{\alpha-1} \right] h_k^i \\ G_c^i &:= \left. \frac{\partial \dot{g}_k}{\partial c_k} \right|_{*i} = \mathcal{K}^i(\cdot) \left[(1+\varepsilon) - x^i(\cdot) \right] \end{aligned}$$

$$G_h^i := \left. \frac{\partial \dot{g}_k}{\partial h_k} \right|_{*i} = -\mathcal{K}^i(\cdot) \left[\tilde{x}_h^i(\cdot) \gamma + x^i(\cdot) \xi \eta_h^i \right]$$

$$G_g^i := \left. \frac{\partial \dot{g}_k}{\partial g_k} \right|_{*i} = \mathcal{K}^i(\cdot) \left[(1 + \varepsilon) \left(1 - \alpha A (g_k^i)^{\alpha-1} \right) - \tilde{x}_g^i(\cdot) \gamma^i - x(\cdot) \left(1 + \xi \eta_g^i - \alpha A (g_k^i)^{\alpha-1} \right) \right]$$

Interestingly, we can observe that when $\eta_0 \rightarrow 0$, $e_t = \eta(e_t) = \Psi^1(\cdot) = \Psi^2(\cdot) = x(\cdot) = 0$ and $\mathcal{K}(\cdot) \rightarrow (1 + \varepsilon)^{-1} \mathcal{M}(g_k^i)|_{\eta \rightarrow 0}$.