A Macro-Financial Model of Monetary Policies with Leveraged Intermediaries[*]

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Matthieu Darracq Paries[†] and Quentin Vandeweyer[‡]

February 12, 2018

Abstract

This article presents a macro-financial model in which leveraged intermediaries play the key role in the transmission of monetary policies. Liquidity frictions in the money market create a role for central bank reserves as the ultimate mean of settlement. The model is able to reproduce and rationalize a series of facts both in normal times and in financial crises: (i) monetary policy can be implemented by manipulating both the interest paid on excess reserves (IOER) and the quantity of excess reserves (ii) a strong correlation between money market stresses and credit spreads (iii) a spiraling doom loop between funding and market liquidity (iv) liquidity injection from central banks alleviating liquidity stresses and (v) asset purchase programmes stabilizing asset prices by extracting and suppressing funding liquidity risk from the market.

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[Money] is a commodity subject to great fluctuations of value and those fluctuations are easily produced by a slight excess or a slight deficiency of quantity. Up to a certain point money is a necessity. If a merchant has acceptances to meet tomorrow, money he must and will find today at some price or other. And it is this urgent need of the whole body of merchants which runs up the value of money so wildly and to such a height in a great panic. On the other hand, money easily becomes a drug, as the phrase is, and there is soon too much of it.

Bagehot (1873, p58)

1   Introduction

The traditional neo-Keynesian approach to monetary economics abstracts from financial and liquidity frictions and to assume a moneyless – pure credit – economy featuring a central bank controlling the short-term nominal rate. Nominal rigidities then ensure that changes in nominal rates affect real rates in the short run, such that monetary policy has an impact on real macroeconomic variables and long-term yields are determined by the expectation hypothesis. This contrasts with the current focus in finance and banking literatures. Empirical finance has showed that movements in asset prices are mainly attributed to changes in the risk premia rather than in cash flows, while the banking literature has been stressing the importance of liquidity frictions in the transmission of monetary policy. In particular, forceful liquidity frictions and resulting liquidity risk have proved be able to generate dramatic drops in asset prices during the subprime crisis (Gorton and Metrick (2012)).

In this paper, we propose a macro-financial model with a leveraged financial sector and a money market subject to friction shocks. The model is able to rationalize both conventional monetary policy operations in normal times and the drop in asset prices when increased funding liquidity frictions interact with market liquidity frictions to create a double liquidity spiral. Under these circumstances, the central bank can stabilize the economy both by injecting reserves into the banking system

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and by directly purchasing securities from the market and, thereby, extracting and supressing liquidity risk.

Overall, in our model, monetary policy affects credit and other real macro-variables by stirring the risk premia on long-term assets. In contrast with the neo-Keynesian view, it is financial and liquidity frictions rather than nominal frictions that empowers monetary authorities. In our model, a leveraged banking sector, that is marginal in both credit and money markets, is the key player in the transmission of monetary policy. More precisely, we assume the existence of liquidity frictions in the money markets that prevents the payment system from clearing every transaction with certainty and therefore creates liquidity risk and a role for central bank reserves. By manipulating both interest paid on and the supply of excess reserves, the central bank impacts the short term money market rates as well as the amount of risk taken by the banking sector.

The model is able to describe key mechanisms observed during the 2007-2008 financial crisis and its recovery. First, introducing funding liquidity frictions depending on the health of the banking sector reinforces the market liquidity spiral as described in Brunnermeier and Sannikov (2014). During a banking crisis, the functioning of private money markets becomes severely impaired (Gorton and Metrick (2012)). As counter-party risk and uncertainty over future access of liquidity escalate, financial institutions increase their demand for short term maturity assets. This *funding* liquidity stress crucially interacts with *market* liquidity stress forcing a fire-sale liquidation of these capital market assets. Once it reaches a certain threshold, bank capital becomes the prey of a two way self-reinforcing liquidity spiral. On the market liquidity side, lower capital forces liquidation which increases endogenous volatility and decreases bank capital even more. On the funding liquidity side, a negative shock to bank capital impairs the functioning of the money market which increases funding liquidity risks and decreases asset prices. The interaction of these two liquidity frictions creates extreme non-linearities which strongly amplifies macroeconomic shocks of moderate size whenever these shocks are concentrated

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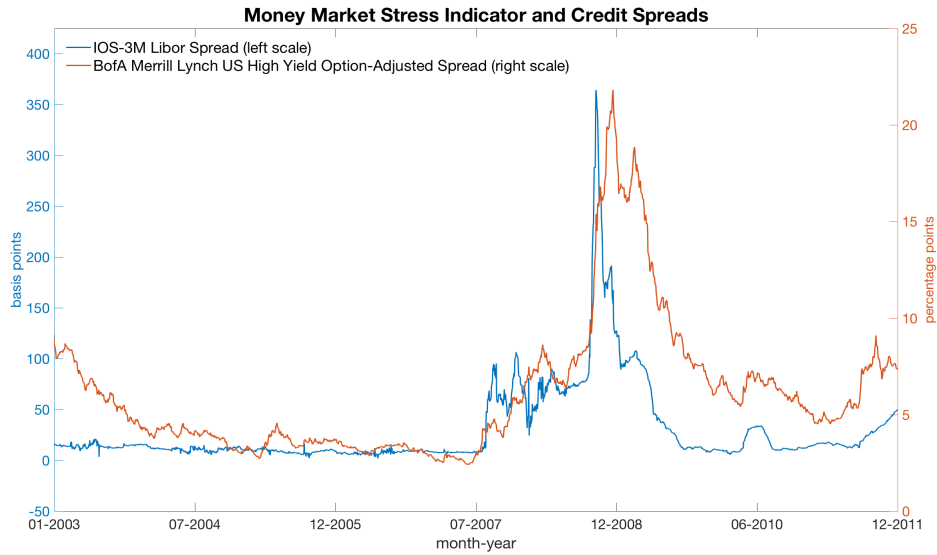
Figure 1: Money Market Stress and Credit Spreads (data from Bloomberg Professional.)

in a part of the economy to which the banking sector is exposed. In this setting, the central bank can directly intervene by pouring enough liquidity to counter the funding liquidity spiral with various liquidity facilities or indirectly remove liquidity risk from private markets by holding long securities in its own balance sheet. We highlight that key assumption for this mechanism to operate is that the central bank is not subject to liquidity risk as it the creator of the ultimate means of settlement. This mechanism mirrors the events of the 2007-2008 when the financial sector was holding a significant amount of mortgage backed securities and was therefore exposed to the subprime housing bubble. In particular, the model rationalizes strong co-movement between three month OIS-Libor spreads—a commonly used indicator of money market stresses—and credit spreads (see Figure 1). Our model rationalizes this relationship as the consequence of feedback from funding liquidity to market liquidity.

Our macro-financial model features an heterogeneous banking sector[1] that is involved in both risk and liquidity transformation. The financial sector holds risky long term claims to the production sector while issuing short term risk free debt to the household sector. By doing so, it exposes itself to credit, interest rate and liquidity risk. Liquidity risk is the risk of having to rollover its debt at a higher rate than the money market rate. In normal times, this liquidity risk is very low as the money markets are functioning properly. Yet, this liquidity risk, even if quantitatively low, is very important for central bank open market operations to affect the short term nominal rate by varying the liquidity premia at which the money markets trade on top of the interest on reserves. The existence of this risk creates a positive demand for central bank reserve for precautionary saving motives which breaks down the Wallace (1981) neutrality result of open market operations.

We first use the model to investigate how monetary policy can be implemented to stabilize inflation in a general equilibrium framework with liquidity frictions by using a combination of the different policy tools with one degree of freedom. Second, we analyze how the model reacts to a large increase in the amount of non performing loans and show that the model consistently reproduce a series of stylized fact observed during the crisis. In particular, the liquidity spiral from Brunnermeier and Sannikov (2014) is reinforced by the disturbances in the money markets such that even a low valorization differential can generate substantial drop in asset prices. Third, we show how, in the model, the central bank can intervene in order to mitigate this liquidity spiral by increasing the supply of excess reserves and purchasing long term risky assets.

Literature Review. Introducing monetary policy into a Walrasian macroeconomic framework is a challenging task commonly referred as the Hahn (1965) problem. The fundamental reason is that, by construction, general equilibrium models feature a benevolent auctioneer that solves perfectly any liquidity friction. This

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[1]Banking sector has to be understood in the larger sense as any financial institutions involved in liquidity transformation which includes the shadow banking sector

challenge is not new and has surfaced in different forms in the history of economics. From the finance side, Black (1970) depicts a world without money in which every transactions is settled by private banks in perfect market; subsequently questioning the very reason of d'etre of central banks. From the macroeconomic side, Wallace (1981) shows that under quite general assumptions, open market operation should not have any impact on the economy; not even on nominal variables. Neo-Keynesian models a la Woodford (2003) do also follow this path of monetary Walrasianism[2] by assuming a moneyless world where the central bank controls the nominal short rate by an arbitrage equation. This work departs from these strands of the literature by assuming the existence of liquidity frictions which create a special role in the payment system for money – in the form central bank reserves – with inspiration from the literature on monetary policy implementation (see Bindseil (2014) for an introduction).

In our model, (inside) money is endogenous and adapts to the liquidity needs of the payment system, relating this paper to earlier work of Gurley and Shaw (1960). In this dimension, this paper is also close to Brunnermeier and Sannikov (2016b). The main distinction between the two articles appears in the function given to money. In their work, money is held by economic agents as an imperfect risk sharing device in a world with incomplete financial markets. We rather stress the role of central bank money as the ultimate means of settlement and focus on the imperfect complementarity between private and central money creation. In our model, the majority of the transactions are cleared by private banking agents ensuring that, in normal times, there is a sufficient elasticity of credit to almost completely washes out these payment frictions. As in Schneider and Piazzesi (2015), we adopt an institutionalist perspective according to which the payment system is layered and hierarchical with a banks using central bank liability as money and households using bank liability as money. We share a similar mechanism according to which a particular monetary policy regime can be implemented with different combinations of policy tools and featuring a regime of reserve satiation but differ in focussing on crisis dynamics by modeling explicitly risk and solving for non-linearities rather than the effect of

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[2]See Mehrling(1998) for a discussion of the concept and its history in economic thought

changes in the supply of collaterals. This work is also closely related to Bianchi et al. (2014) who similarly introduce a payment system with frictions to create a special role for central bank money in a macro model. We differ by adopting a more stylized approach to liquidity risk in order to focus on non-linear crisis effects that arises in the general equilibrium.

Last, our modeling strategy is based on the growing literature in macro-finance modeling with a financial sector (Brunnermeier and Sannikov (2014), He and Krishnamurthy (2013), Di Tella (Forthcoming), Silva (2015a)). In particular, our model follows Drechsler et al. (Forthcoming) in assuming that liquidity risk in maturity transformation creates an additional cost to bank leverage which is embedded in asset prices. Moreover, in order to build a layered payment system, we follow Di Tella and Kurlat (2016), Drechsler et al. (2016) and Klimenko et al. (2016) in including bank deposits in the utility function of households as a proxy for transaction services.
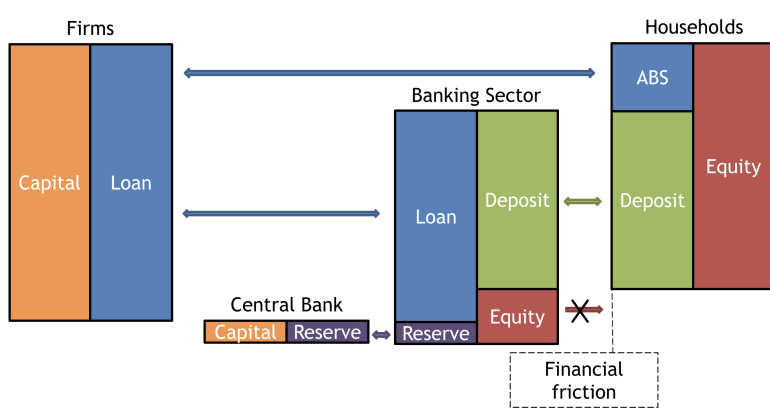
2 The Model

Outline

The model is an infinite-horizon stochastic production economy in continuous time with heterogeneous agents and financial frictions. There are three sectors in the economy: the productive sector, the household sector and the banking sector. The productive sector has access to a constant return to scale production technology and finances its physical capital holdings by issuing corporate bonds held by both banks and households as perfectly diversified Asset Back Securities (ABS). With a given probability, firms default on these bonds. The household sector maximizes its utility by consuming and holding its wealth both in ABS and bank deposits. We capture the transaction services derived from holding deposits by assuming that these deposits feature in households utility function. The banking sector lies at the intersection of the household and production sectors. With respect to the household sector it possesses the two comparative advantages of having a better monitoring

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technology that reduces the default probability of its loan holdings and of being able to issue deposits that are used by households for their transactional properties. In order to exploit these advantages, the banking sector is simultaneously active in the businesses of risk and liquidity transformation. By doing so, it exposes itself to the three fundamental risks:

- **Liquidity Risk.** Issuing on-demand deposits to the household sector that are used for transaction purposes exposes individual banks to a potential temporary funding gap. The payment system requires this individual bank to fill this gap before the end of the day either by acquiring inter-bank lending from another bank with funding surpluses or by resorting to the Central Bank discount window. In a frictionless economy where deposits cannot be converted into cash, this liquidity risk would be zero has there would always be another bank in surplus willing to lend its fund at the market rate. Yet, just a little bit of uncertainty in the bank's ability to acquire the given funds at the average of the inter-banking market rate by the end of the day generates a residual liquidity risk that is sufficient to create a demand for excess reserves.

- **Interest Rate Risk.** Because banks are holding long term securities while issuing short term deposits, a change in the market real interest rate (for instance due to a change in preferences or productivity), it will suffer a loss of real net worth.

- **Credit Risk.** By holding defaultable securities, banks also expose themselves to credit risk. The relevant macroeconomic definition for credit risk is an exposure to an unexpected increase in the proportion of non performing loans.

These risks are linked to the three financial frictions of the model. The first two ones are similar to Brunnermeier and Sannikov (2014). First, the financial sector cannot issue equity to the household sector (or equivalently, macro-risks are not tradable in financial markets) which prevents optimal risk distribution across agents. Second, there is a technological illiquidity in bond issuance or repurchase akin to a capital adjustment cost. Last, as in Bianchi et al. (2014), we assume that in the

Figure 2: Model Balance Sheets

interbank money market participants do not match with certainty which creates a role of precautionary holdings of excess reserves. Figure 2 provides a sketch of the balance sheet of the private agents in the model.

Demographics

We denote variables referring to the household sector as underlined to distinguish it from bankers. For instance, the wealth of a household j is denoted as $\underline{n}_{jt}$ and of a banker i as n_{it}. We write aggregate variables by removing individual indices: $n_t = \int_i n_{it}$ and $\underline{n}_t = \int_j \underline{n}_{jt}$. As in Brunnermeier and Sannikov (2014), the main state variable for our model is the net worth – capitalization – of the banking sector relative to the size of the economy. We write this state variable as $\eta_t = \frac{n_t}{n_t + \underline{n}_t}$. To ensure stationarity, we assume that the economy follows a continuous time OLG structure a la Gârleanu and Panageas (2015) and Drechsler et al. (Forthcoming). Agents die at rate κ and new agents are also born at a rate κ with a fraction $\bar{\eta}$ as bankers and a fraction $1 - \bar{\eta}$ as households. The law of motion of η can therefore be written as:

$$d\eta_t = \kappa(\bar{\eta} - \eta_t)dt + \eta_t(1 - \eta_t)\left[\mu_{\eta,t}dt + \sigma_{\eta,t}dZ_t\right] \tag{1}$$

Where $\mu_{\eta,t}$ and $\sigma_{\eta,t}$ is the part of the law of motion of wealth that evolves accord-

ing to the difference of returns between to two types of agets. It has to be determined endogenously according their respective portfolio choices.

Technology and Asset Backed Securities Market

Atomistic risk-neutral firms have access to a constant return to scale technology production function $y_t = ak_t$. Markets are competitive and firms always operate with zero net worth. They finance all physical capital holdings with loans at a fixed rate $r^l = a$ such that any profit made by the firm is entirely used to repay interests on loan. As in Leland and Toft (1996), we assume that loans have geometrically decreasing annuity such that the loan depreciate at rate δ. Moreover, we assume that firms default on their loans with probability p_t under time increment dt and that this probability follows the diffusion process: $dp_t = \sigma dZ_t$ where $Z = \{Z_t \in \mathbb{R}^d; \mathcal{F}_t, t \geq 0\}$ is a standard adapted Brownian motion. From the point of view of an investor (a bank or a household), a firm is therefore simply an investment technology. All loans are bundled into a perfectly diversified ABS asset l_t. We write the law of motion of the stock of ABS holding for the two types of investors as:

$$\text{Banker:} \quad \frac{dl_t}{l_t} = [\Phi(\iota_t) - (\delta + p_t)] \, dt + \sigma dZ_t$$

$$\text{Household:} \quad \frac{d\underline{l}_t}{\underline{l}_t} = [\Phi(\underline{\iota}_t) - (\delta + p_t + \epsilon)] \, dt + \sigma dZ_t$$

The loan issuance function $\Phi(\iota_t)$ transforms $\iota_t l_t$ real output into Φ more loans per time increment. It takes into account that issuing loans takes increasingly more resources and therefore has negative return to scale, $\Phi'_t > 0$ and $\Phi''_t \leq 0$. This assumption reflects frictions in the issuance of loans akin to capital adjustment cost with decreasing return to scale[3]. Parameter δ is the rate at which loans matures while

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[3]This assumption can be rationalized if screening the quality of the borrowers is resource consuming or if the pool of quality borrowers is scarce. This assumption is important for two different reasons. First, because of the production technology is AK, without this adjustment cost, the optimal supply of loans would be infinite. Second, this feature creates some downward rigidity such that it is increasingly costly for the bank to buy back a loan that was granted to a firm. This

p_t is the default rate under time increment dt and ϵ reflects the higher proportion of default that household investors face as compared to banks with a higher expertise. As the economy only features one aggregate stochastic process dZ_t, we can postulate that the stochastic law of motion of the price of a unit of ABS q_t follows:

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t$$

where μ_t^q and σ_t^q are to be determined endogenously in general equilibrium conditions. The flow of return on loan holdings is therefore respectively:

$$\text{Banker:} \quad dR_t = \underbrace{\left(\frac{r^l - \iota_t}{q_t} + \Phi(\iota_t) - (\delta + p_t) + \mu_t^q + \sigma\sigma_t^q\right)}_{\mu_{R,t}} dt + \underbrace{(\sigma + \sigma_t^q)}_{\sigma_{R,t}} dZ_t$$

$$\text{Household:} \quad d\underline{R}_t = \underbrace{\left(\frac{r^l - \underline{\iota}_t}{q_t} + \Phi(\underline{\iota}_t) - (\delta + p_t + \underline{\epsilon}) + \mu_t^q + \sigma\sigma_t^q\right)}_{\mu_{\underline{R},t}} dt + \underbrace{(\sigma + \sigma_t^q)}_{\sigma_{\underline{R},t}} dZ_t$$

The first term of the drift is the net dividend price ratio of holding a unit of securitized loan in your book after new issuance. The remaining part is the capital gain and the loading factor is the total volatility of the return process which consist in the sum of the exogenous volatility of probability of default of loans and the endogenous volatility due to general equilibrium responses in prices.

Banking Sector and Interbank Money Market

The banking sector differs from the household sector for two reasons. First, it can create a particular type of liability that is used by households for transaction purposes[4]. Second, when held by a bank, a smaller proportion of ABS defaults. In order

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assumption is therefore necessary for under-capitalization of the banking sector to generate fire-sale pressure.

[4]We do not model explicitly the usage of deposits in the payment system but capture it by assuming that it creates liquidity risk in banks' balance sheet and affects positively the utility of households

to exploit its advantages in holding ABS and issuing deposits, the banking sector leverages by financing illiquid long-term risky loans with short term (on demand) liabilities. By doing so, the banking sector becomes exposed to liquidity risk: the excess volatility in revenue created by this liquidity management problem. Bankers maximizes their life-time expected logarithmic utility:

$$\max_{\{w_t, \iota, \hat{c}_t\}_t^\infty} E_t \left[\int_t^\infty e^{\rho\tau} \log(\hat{c}_\tau n_\tau) d\tau \right] \tag{2}$$

s.t.

$$
\begin{aligned}
\frac{dn_t}{n_t} =& \left[i_t + w_t(\mu_{Rt} + \pi_t - i_t) + w_t^r(i_t^r - i_t) + w_t^d(i_t^d - i_t) - \hat{c}_t - \pi_t \right] dt \\
&+ w_t \sigma_{Rt} dZ_t + \frac{dT}{n_t} + \chi(w_t^r, \theta_t) \min(w_t^d, 0) d\tilde{Z}_t
\end{aligned}
\tag{3}
$$

The first part of the law of motion for the wealth of banks is the traditional Merton problem where, i_t is the nominal[5] risk-free interest rate, w_t the portfolio weight on ABS, w_t^d and i_t^d respectively the portfolio weight and the nominal interest paid on deposits and $\hat{c}_t$ the consumption rate. Banks also receives a flow of transfers per unit of wealth of $\frac{dT}{n_t}$ from the central bank to the banking sector. To this problem, we add both a new (liquidity) friction and a new (reserve) asset. The new friction is captured by the last term. Banks are subject to an idiosyncratic adapted Brownian process $\tilde{Z} = \{\tilde{Z}_t \in \mathbb{R}^d; \mathcal{F}_t, t \geq 0\}$ which increases the volatility of the stream of revenue of banks proportionally to its leverage (i.e. when $w^d < 0$) according to the size of the $\chi(w_t^r, \theta_t)$ function. This function captures the imperfection of money markets in a context where the maturity mismatch of the bank balance sheet creates short term funding gaps. In Appendix C, we discuss the micro-foundation of the idiosyncratic liquidity Brownian process as a restriction on the specification of Bigio and Bianchi (2017) and Afonso and Lagos (2015). The intuition for this term is that market imperfection creates fluctuations in the actual rates at which banks finance themselves in the money market around the average rate of the money market. The parameter θ_t controls the intensity of the liquidity frictions and is therefore

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[5]We assume that consumption good prices evolves deterministically as $\frac{dp_t}{p_t} = \pi_t dt$.

an indicator of money market perfection. Central bank reserves can be held by banks with a portfolio weight w_t^r in order to mitigate this liquidity risk as they can serve as a buffer in case of funding gap. On these balances, the central bank pays nominal interest i_t^r, the interest paid on excess reserves (IOER). We capture these two feature by assuming that liquidity risk is a decreasing function of both the amount of reserves in the portfolio of the bank $\partial \chi(w_t^r, \theta_t)/\partial w_t^r < 0$ and increasing in the frictions in the money markets $\partial \chi(w_t^r, \theta_t)/\partial \theta_t < 0$. Finally, we also define for later use a corresponding real rate for every nominal rates: $r_t = i_t - \pi_t$, $r_t^r = i_t^r - \pi_t$ $r_t^d = i_t^d - \pi_t$ where π_t is the inflation rate.

Household Sector

The household sector maximizes its logarithmic life-time utility function which includes deposits (the liability issued by banks) in their utility function:

$$\max_{\{\underline{w}_t, \hat{\underline{x}}_t, \underline{\iota}_t\}_t^\infty} E_t \left[\int_t^\infty e^{-\rho\tau} \log(\hat{\underline{x}}_\tau \underline{n}_\tau) d\tau \right] \tag{4}$$

s.t.

$$\frac{d\underline{n}_t}{\underline{n}_t} = \left(i_t^d + \underline{w}_t \left(\mu_{\underline{R},t} + \pi_t - i_t^d \right) - \hat{\underline{c}}_t - \pi_t \right) dt + \underline{w}_t \sigma_{\underline{R},t} dZ_t + \frac{d\underline{T}}{\underline{n}_t} \tag{5}$$

$$\hat{\underline{x}}_t \underline{n}_t = \hat{\underline{c}}_t^\beta \left(\underline{w}_t^d \right)^{1-\beta} \underline{n}_t \tag{6}$$

Where $\hat{\underline{x}}_t \underline{n}_t$ captures the non-pecuniary valorisation of deposits by households with a Cobb-Douglas aggregator between consumption and deposits with parameter β in (6) reflecting their substitutability in the utility function of household agents. Otherwise, the programme is similar to the one for banks with the exception that, because households do not issue deposits[6], they do not expose themselves to the idiosyncratic liquidity risk.

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[6]We make this assumption without loss of generality as in equilibrium banks are issuing deposits to households

Central Bank

The law of motion of central bank's net worth is given by:

$$dn_{cb,t} = \left(-M_t i_t^r + L_{cb,t} \mu_{R_{cb},t} \right) dt + L_{cb,t} \sigma_{R_{cb},t} dZ_t - (dT_t + d\underline{T}_t) \tag{7}$$

On top of the policy rates, the central bank can potentially decide to purchase ABS by setting $L_{cb,t} > 0$ and therefore getting exposure to the aggregate macroeconomic risk and receives the return:

$$dR_{cb,t} = \underbrace{\left(\frac{r^l - \iota_t}{q_t} + \Phi(\iota_t) - (\delta + p_t + \epsilon_{cb,t}) + \mu_t^q + \sigma\sigma_t^q \right)}_{\mu_{R_{cb},t}} dt + \underbrace{(\sigma + \sigma_t^q)}_{\sigma_{R_{cb},t}} dZ_t$$

We assume that the central bank is not as efficient as banks in managing ABS and therefore face higher losses than banks but still less than households. Importantly, the central bank, being the issuer of reserves, does not face idiosyncratic liquidity risk. To keep the model tractable and be able to focus on its features of interest, we make the following assumption regarding the balance sheet of the central bank:

Assumption 1 (Scalability of monetary policy operations)**.**

- dT and $d\underline{T}$ are set at each point in time such that central bank wealth is constant $dn_{cb,t}/n_{cb,t} = 0$ and the distribution of wealth between banking and households sector η_t is not impacted directly by policy decisions $\{m_t, L_{cb,t}, i_t^r\}$ taken in combination.

- The proportion of defaulting loans when held by the central bank is equal to the average default rate in the economy at time t $\epsilon_{cb,t} = \psi_t \epsilon$ where $\psi_t = w_t n_t / (w_t n_t + \underline{w}_t \underline{n}_t)$ is the share of total loans held by the banking sector such that aggregate return is unnaffected by monetary policy.

This assumption ensures that, whatever its monetary policy stance, the central bank won't affect the distribution of wealth between agents nor the aggregate return. Consequently, it allows us to focus on the effect of changes in the central bank's balance sheet size and policy rates without having to explicitly model the impact of open market operation on the balance sheet of the central bank.

3   Solving the Model

Optimal Loan Issuance

Given the postulated dynamics for the price of securitized loans q, the problem of choosing the optimal issuance rate ι_t for any investor can be separated from the rest of the dynamical problem. Optimal investment rate is given by:

$$\Phi'(\iota_t) = \Phi'(\underline{\iota_t}) = \frac{1}{q_t} \tag{8}$$

Which equals the marginal benefit of loan issuance on return to it's marginal cost for consumption today at current price q_t. Because Φ is a concave function, issuance rate is a positive function of q_t.

Banking Sector

The problem has the form of classical leveraged portfolio decision problem with risk-free liabilities generating excess liquidity risk when the agent is leveraged. Applying the optimality principle to the problem, we find the following first order conditions:

$$\text{Consumption:} \quad \hat{c}_t = \rho \tag{9}$$

$$\text{Securities:} \quad \sigma_{n,t}(w_t) = \frac{\mu_{R,t} - r_t}{\sigma_{R,t}} \tag{10}$$

$$\text{Deposits:} \quad -w_t^d = \frac{r_t - r_t^d}{\chi_t^2} \tag{11}$$

$$\text{Reserves:} \quad (r_t^r - r_t) = (-w_t^d)^2 \chi(w_t^r, \theta) \chi_{w^r}(w_t^r, \theta) \equiv -s(w_t^r, \theta_t) \tag{12}$$

The first two equations are standard: the optimal consumption rate is constant and equal to the time discount rate while the portfolio weight on risky asset holding is such that the volatility of the stochastic discount factor equals the Sharpe ratio. The third equation is such that deposit funding $-w_t^d$ is a positive function of the spread between the deposit rate r_t^d and the money market rate r_t but also a negative function of the liquidity risk scaling factor. Because leveraging increases exposure to liquidity risk, optimal deposit issuance decreases whenever this risk increases. The last equation is the asset pricing condition in the market for central bank reserve. The equation states that the spread between the money market rate and the interest paid on reserves (i.e. the effective opportunity cost of keeping reserve for a time increment) must be equal to the marginal benefit of holding reserve balance in decreasing the volatility of the income stream[7]. This effect depends positively on bank leverage $-w_t^d$, the level of liquidity risk χ_t and the marginal reduction in the liquidity risk scaling factor due to increasing reserve holdings. For later use, we define the relationship that ties the spread between the money market risk free rate and the rate paid on reserve with the quantity of reserves and the money market tightness through (12) as $(r_t - r_t^d) \equiv s(w_t^r, \theta_t)$.

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[7]In reality there also exists another benefit in expectation generated by the gap between the interest paid at the discount window and the one at the deposit facility. The implementation literature usually assume risk neutral banks and focusses on this feature. We abstract from it for tractability reasons but adding it would not qualitatively change our results.

Households' Choices

The problem has the form of a Merton problem with the risk-free asset providing non-pecuniary services. we find the following first order conditions:

$$\text{Consumption:} \quad \hat{\underline{c}}_t = \rho\beta \tag{13}$$

$$\text{Securities} \quad \sigma_{\underline{n},t}(\underline{w}_t) = \frac{\mu_{\underline{R}t} - r_t^d}{\sigma_{Rt}} - \frac{\rho(1-\beta)}{\sigma_{Rt}(1-\underline{w}_t)} \tag{14}$$

Both equations provide the classical results from the Merton problem with logarithmic agents adjusted for the deposit in the utility function feature. Households therefore always consume a fixed portion $\beta\rho$ of their net worth. With higher preference for liquidity, a lower β, agents will want to consume less to hold more deposits. Moreover, because with logarithmic utility, substitution and wealth effects exactly cancels each other out, the optimal portfolio weight is composed of the myopic demand and a term taking into account the direct benefit of holding deposits for their direct impact on utility. Equation (14) is quadratic with the optimal solution given by its lower root. Note that whenever β converges to one, i.e. households do not value the liquidity services of deposits, the FOCs converges to the solution of the classical Merton problem.

Equilibrium

Definition 2 (Equilibrium definition). *Given an initial allocation of all asset variables at $t = 0$, an equilibrium is a set of adapted stochastic processes for the interest rates $\{i_t : t \geq 0\}$, $\{i_t^d : t \geq 0\}$, asset prices $q_t : t \geq 0\}$, loan holdings $\{w_t : t \geq 0\}$, $\{\underline{w}_t : t \geq 0\}$, loan issuance rate $\{\iota_t : t \geq 0\}$, aggregate loan stock $\{L_t : t \geq 0\}$, consumption rates $\{\hat{c}_t : t \geq 0\}$ and $\{\hat{\underline{c}}_t : t \geq 0\}$, and inflation process $\{\pi_t : t \geq 0\}$ such that:*

1. *Markets for ABS, interbank lending, reserves and consumption goods clear,*

(a) *ABS:* $\eta_t w_t + (1 - \eta_t)\underline{w}_t = q_t L_t - L_{cb,t}$

(b) *Interbank lending:* $w_t^{ib} = 0$

(c) *Reserves:* $w_t^r = \bar{m}_t$

(d) *Output:* $\eta_t \hat{c}_t + (1 - \eta_t)\underline{\hat{c}}_t = y_t/n_t$

(e) *Deposits:* $\eta_t w_t^d + (1 - \eta_t)\underline{w}_t^d = 0$ *(clears through Walras law)*

2. *Households solve problem* (4)*, Banks solve problem* (2)*,*

3. *Central bank sets reserve money supply* $\{\bar{m}_t : t \geq 0\}$*, interest rate paid on reserve* $\{i_t^r : t \geq 0\}$ *and security holdings* $\{L_{cb,t} : t \geq 0\}$

4. *The process for money market perfection* θ_t *is given exogenously*

As in Brunnermeier and Sannikov (2014), the model is solved as a recursive equilibrium in one state variable $\eta_t = n_t/(n_t + \underline{n}_t)$ the capitalization of the banking sector relative to the total size of the economy.

Our objective with this paper is to provide some qualitative insights about the mechanisms driving liquidity crises and central bank interventions rather that quantitative ones. We therefore calibrate our variables loosely in order to be as close as possible to the numerical exercice in Brunnermeier and Sannikov (2016a). We use the following default functional form for the loan issuance function: $\Phi(\iota) = \log(\kappa\iota + 1)/\kappa$ and numerical value for parameters: $\kappa = 10, \rho = 6\%, a = 13\%, \sigma = 10\%, \epsilon = 3\%, \delta = 3\%$. We solve the model numerically with a finite difference approach described in the online appendix.

4   The Normal Regime

In this section, we analyze the dynamics of the model in its non-crisis locus, i.e. whenever banks are well capitalized and the interbank money market is functioning with few frictions. For this purpose, we will consider for this section that the parameter of money market relaxation remains fixed to a high level $\theta_t = \bar{\theta} = 1 - \varepsilon$

with ε being arbitrarily small. We make the second assumption that the central bank is able to impose a ceiling on the money market rate r_t by allowing a subset of financial institutions to access overnight loan of reserves at a discount window rate r_t^{dw}. This bound is extended for all the money market because well capitalized banks with access to the discount window could arbitrage any rate above it. We will release both assumptions in the next section.

Conventional Monetary Policy

Money market imperfection combined with the assumption that central bank reserves can be held in order to mitigate these frictions create a special role for the central bank as the supplier of this reserve asset[8]. The ability the central bank has to affect both the price and the quantity of reserve creates an under-determination of the system.

Proposition 3 (Monetary policy has one degree of freedom)**.** *The central bank controls both the supply of excess reserve available to banks $w_t^r = \bar{m}_t$ and the nominal interest rate it pays on reserves $i_t^r = \bar{i}_t^r$. The system is therefore under-determined. All proofs are relegated to appendix A.* □

This can be seen from combining the Fischer equation $i_t = r_t + \pi_t$ and the asset pricing condition for reserve (12) to find:

$$\pi_t = [s(\bar{m}_t, \theta_t) + \bar{i}_t^r] - r_t$$

Inflation is determined as the deviation between the nominal interest rate prevailing in money markets, which equals the interest rate paid on reserve plus the money market spread. As the money market spread depends on the amount of central bank reserve supplied to banks at time t, the central bank can affect this deviation with

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[8]The supply of reserves can be thought of as being implemented through open market operations. For tractability, we do not model these operations explicitly although the model would be similar with an instantaneous risk free treasury bill asset held only by banks that would be swapped by the central bank at market prices

two different policy tools: interest paid on reserves and manipulation of the quantity of reserves. In other words, the central bank has one degree of freedom in its implementation strategy.

For example, until 2011, the Federal Reserve was not providing a deposit facility to excess reserves, implicitly setting $\bar{i}_t^r$ to zero. Every adjustment in the monetary policy stance was therefore taking place as a shift in the liquidity spread implemented by daily adjustment in the supply of reserves through repo operation. In our framework, this translates in adjustment in $\bar{m}_t$ in order to affect $s(\bar{m}_t, \theta_t)$[9]. Conversely, the European Central Bank since its establishment has been following a symmetrical corridor operational framework. Under this regime, the ECB sets the bounds of the corridor at a fixed 200 basis point spread and adjusts the reserve supply in order for the spread to clear half ways at 100 basis point. In this case, the ECB implements its monetary policy stance effectively by shifting the interest it pays on excess reserves $\bar{i}_t^r$ (deposited at the ECB) rather than moving the spread $s(\bar{m}_t, \theta_t)$. By Proposition 3, these two different ways of implementing monetary policy yield an exact same monetary policy stance.

In our model, we assume that the supply of reserves $\bar{m}_t$ is set to keep $s(\bar{m}_t, \theta_t)$ at a fixed but positive level and inflation targeting is implemented by shifting $\bar{i}_t^r$[10]. Note that because the model economy does not feature nominal rigidities, it is ruled by a *Fisherian* relation and monetary policy has the *wrong sign*. We abstract from these considerations by simply assuming that in any case the central bank will stabilize the inflation to zero by making sure that the money market nominal rate i_t equals

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[9]In 2006, the Fed was given the authorization to pay interest on reserve starting in 2011, which would have given the Fed the opportunity to shift to symmetrical corridor framework. The Fed eventually ended up starting paying IOER much earlier than foreseen as a mean to keep market rates above zero while pouring as much liquidities in the market to fight the burst of the subprime bubble.

[10]In practice, keeping liquidity spread constant requires constantly adjusting the supply of reserve because of the high volatility of what operational literature calls "autonomous factors". Absent these adjustment, these feedbacks into volatility in daily money market rates without any relation with actual macroeconomic development as experience during the Volcker experiment.

the natural real interest rate r_t by moving the interest it pays on reserves:

Assumption 4 (Separation principle). *Inflation is stabilized by the central bank to zero, exclusively by pinning down the interest paid on reserves such that:*

$$\bar{i}_t^r = r_t - s(\bar{m}_t, \theta_t)$$

This assumption reflects the practice at the beginning of the crisis of the *separation principle* according to which the overdetermination of the monetary policy toolbox allows to have IOER focused on maintaining price stability while the quantity of reserves can be adjusted independently to alleviate liquidity stresses in the interbank market (see for example Clerc and Bordes (2010)). Moreover, for tractability, we make a second assumption regarding the following functional form for the liquidity risk scaling parameter $\chi(w_t^r, \theta_t)$.

Assumption 5 (Functional form for liquidity risk). *The liquidity risk scaling parameter $\chi(m_t, \theta_t)$ has the functional form:*

$$\chi(w_t^r, \theta_t) = \max\left\{(1 - \theta_t)(\bar{\chi} - \nu w_t^r), 0\right\}$$

This functional forms has three main characteristics. First, the variable $\theta_t \in \{0, 1\}$ is an index of market perfection such that when the money market is frictionless ($\theta_t = 1$), liquidity risk disappears ($\chi_t = 0$). Second, χ_t is a linearly decreasing function on the portfolio weight on reserve w^r up to a point after which χ is set to zero. We interpret this point as the liquidity satiation threshold.

Definition 6 (A liquidity satiation threshold). *A liquidity satiation threshold is a limit quantity of reserves supply after which any increase does not not reduce exposure to liquidity risk anymore from holding more reserves.*

Figure 3 shows how shifting the quantity of supply of reserves can affect the spread between the real interest paid on reserves and the real money market rate with the given function form. As long as there are non-pecuniary benefits of holding excess reserve, a change in its supply will affect the short term money market rate. In other words, Wallace neutrality holds only after the liquidity satiation has been reached.

Proposition 7 (Conditionnal Wallace neutrality). *A change in the quantity of central bank reserves in circulation w_t^r affects the nominal interest rate if and only banks are not liquidity satiated $w^r < \frac{\bar{\chi}}{\nu}$* ☐

Note that if after the liquidity satiation threshold has been reached, and Wallace Neutrality holds, it doesn't mean that the monetary authority is necessarily powerless as it can still change interest paid on reserves. Overall, monetary policy can be implemented under three different policy regimes which all have been or are currently used by some central banks. The upper panel of fig. 3 shows an economy where the money market rate is implemented in a corridor by adjusting the supply of reserves such that it meets the demand in the downward slopping locus exactly at the policy rate. The middle panel shows an economy where the central bank engineers a structural liquidity deficit such that the money market rate is always at the discount window rate. The lower panel displays a structural excess regime where the money market rate is always pushed to the IOER floor. This corresponds to the situation most mature economies have been in since 2008 and provides a clear intuitition of why the corresponding amount of excess reserves that was created as a side product of quantitative easing policy is not inflationary.

Neutrality of Unconventional Monetary Policy

The central bank can also decide to directly purchase ABS from the market by increasing $L_{cb,t}$. Yet, whenever money markets are functioning properly and liquidity risk is non-existent, these purchases from having any impact on the equilbrium. The reason for this is similar to Silva (2015b), because households and banks are
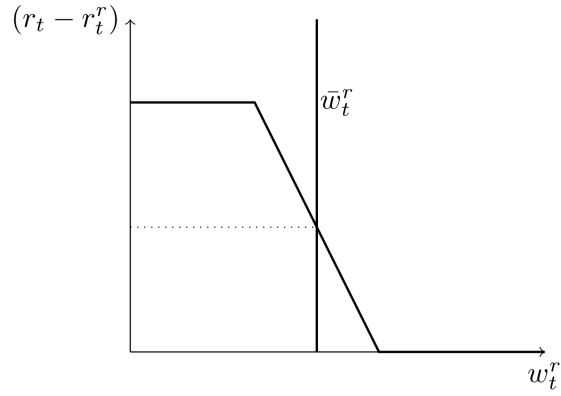
(a) Corridor Policy Regime
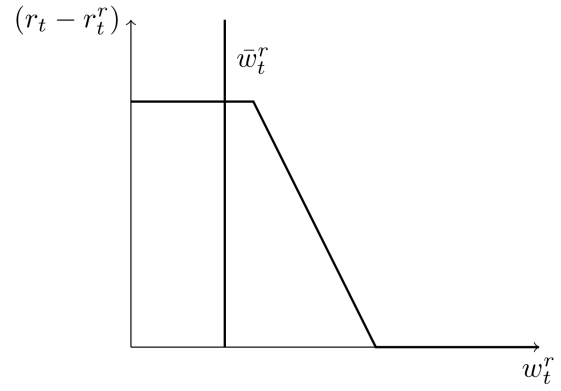
(b) Structural Liquidity Deficit Regime
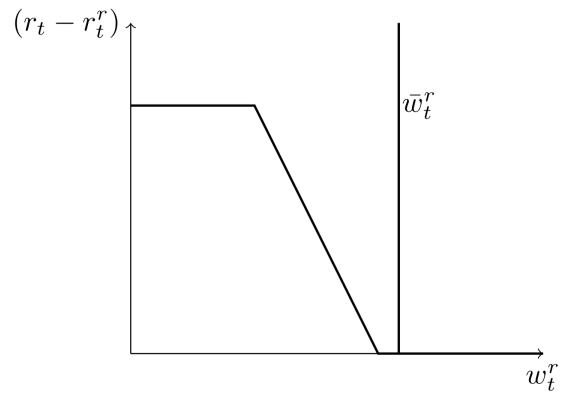
(c) Structural Liquidity Surplus Regime

Figure 3: Money Market Spread over IOER under Different Monetary Policy Regime

exposed to the risk taken by the central bank through transfers[11], and assumption 1 prevents these purchases to have any impact on the wealth distribution. Agents then completely adjust their portfolio demand such that asset prices and real variables are unaffected by these purchases.

Proposition 8 (Outright security purchases from the central bank have no effect on equilibrium in the absence of liquidity stress)**.** *Assume that Assumption 1 holds and $\theta = 1$ so that there is no liquidity risk. Then, for any $L_{cb,t}$, equilibrium variables $\{r(\eta_t), r^d(\eta_t), q(\eta_t), \iota(\eta_t), \hat{c}(\eta_t), \underline{\hat{c}}(\eta_t)\}$ are similar.*

The result of proposition 8 is a of the same nature of the Wallace Neutrality or the Ricardian Equivalence for risk taking. Perfect foresight rational agents are aware that any exposure the central bank is taking on its own balance sheet is actually some risk to which they are themselves exposed. They will therefore adjust their own demand to changes in central bank balance sheet risk such that their net exposure, including the one taken by the central bank, remains the same. Because of assumption 1, there is no redistributive effect of monetary policy and therefore the central bank purchases are completely neutral for the economy.

Risk Taking Behavior

As in Brunnermeier and Sannikov (2014), the existence of pecuniary externalities is such that banks do not internalize their effects on asset prices and take on socially excessive amount of risk. The reason for this is that, because of financial market incomplenetess, banks have an incentive to leverage in order to exploit their technological advantages. i.e. their better loan monitoring technology and bank deposits being in the utility function of the household sector. Incomplete financial markets prevents banks from issuing equity or derivatives, in order to benefit from the technology while offloading the risk to other agents.

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[11]The reality counterpart of these transfers works through the commitment of the fiscal institution to recapitalize banks in case of negative shocks and through central bank profits being rebated to the fiscal authority
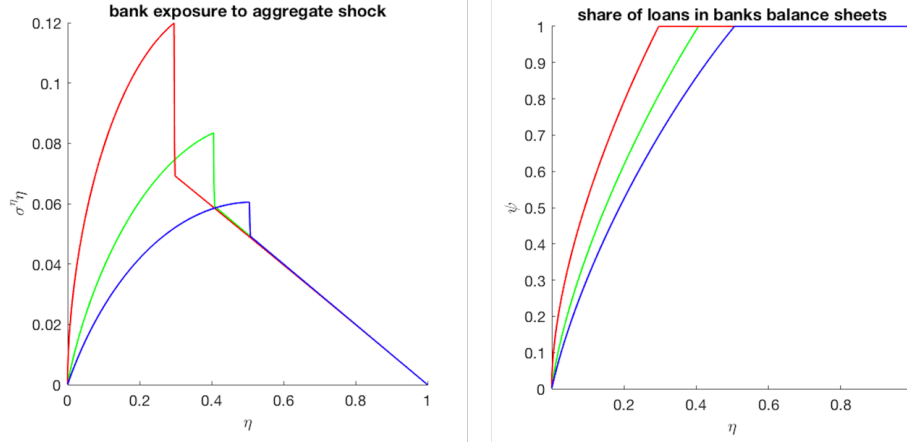
Figure 4: Bank Risk Taking: *blue line is the baseline case with no preference for liquidity $\beta = 1$ and $\epsilon = 0.03$, the middle case in green with $\beta = 1$ and $\epsilon = 0.06$ and the red line with $\beta = 0.95$ and $\epsilon = 0.06$*

Proposition 9 (Bank risk taking)**.** *Risk taking is a positive function of household preference for liquidity parameter $1 - \beta$, banks' relative edge in the monitoring technology ϵ.* □

Figure 4 shows the comparative statics of the trade-off that banks are facing in their leverage decisions. The left-hand side panel shows the banking sector exposure to a shock to aggregate default probability while the right hand side panel displays the leverage of the banking sector. For a higher preference for liquidity $1 - \beta$, the banking sector leverage is higher and more exposed to aggregate risk as the demand for deposit (14) is higher. Similarly, whenever the default parameter differential $\underline{\epsilon}$ is higher, households demand for ABS holding is lower and they therefore rebalance their portfolios demand towards risk-free deposits in order to restore their optimal risk exposure.

5 The Liquidity Crisis Regime

Increased Amplification

Modern liquidity crises have both a market and funding dimension which interacts with each other. Large shocks to their capital force banks to sell their assets to the household sector acting as a second best holder a la Kiyotaki and Moore (1997). These fire-sales negatively affects market valuation, pushing asset prices downwards and endogenous volatility upward. This is the fire-sale spiral from Brunnermeier and Sannikov (2014). During the subprime crisis, this mechanism was also reinforced by a funding dimension. Low bank capitalization also hinders the functioning of money market and affects the ability of the banks to fund themselves in money market. Unsecured money market cease to operate. Access to the interbank market can be completely shut down for the most vulnerable banks. The term structure of money market lending gets steeper reflecting an increase in the value of securing funding for as long as possible (Gorton and Metrick (2012)). Whenever the payment system is disturbed, market participant starts to worry that they won't be able to access money in due time with potentially significant costs even when they are still solvent. In our model, this amounts to connecting money market functioning and liquidity risk to the capitalization of the banking sector, our state variable η_t and removing the discount window floor[12]. With a well capitalized banking sector, there is almost no frictions and liquidity risk tends to zero. As emphasized by Gorton and Metrick (2012), there a is a threshold of stress in terms of bank capitalization and asset prices such that the unsecured money market completely shuts downs and the value of collateral becomes greatly reduced while its volatility increases. The pool of available collateral becomes scarcer as haircuts increase in the repo market. In our model this translate into a negative relation between the functioning of the money market θ_t and the volatility of the price of ABS σ_t^q due to the mechanism of spiraling haircuts described in (Brunnermeier and Pedersen (2009)). We pick up relationship by assuming a logistic functionnal form for θ_t:

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[12]The rationale for doing so is that, many financial institutions do not have access to the discount window it and are too risky to benefit from a loan from the banks having access

$$\theta_t(\eta_t) = \frac{\lambda_1}{1 + \lambda_2 \exp[-((\lambda_3 \eta_t) - \lambda_4)]} + \lambda_5$$

We choose the parameters of this function in order for the money market stresses to start to in the locus where the banks start to fire sale assets to the housing sector (the crisis regime in Brunnermeier and Sannikov (2014)). The conditions in money markets can affect asset prices because, by leveraging to create liquidity services, the banking sector gets exposure to liquidity risk.

Proposition 10 (Liquidity risk transmission mechanism). *Assuming that $\mu_{R,t} - r_t^d \leq (1 - w_t^r)\sigma_{R,t}^2$, an increase in the scaling factor of the liquidity risk χ_t decreases asset prices q_t.* $\qquad\square$

In order to understand this proposition, we start by combining the balance sheet identity $w_t^{ir} + w_t + w_t^r + w_t^d = 1$, the zero supply market clearing condition for interbank lending, (10) and (11) in order to find the following expression for the portfolio weight of the banking sector.

$$w_t = \frac{\mu_t - r_t^d + \chi_t^2(1 - w_t^r)}{\sigma_{Rt}^2 + \chi_t^2} \tag{15}$$

There are two adverse effects of an increase of liquidity risk to the banking sector demand for ABS. First, note that overall risk of leveraging, i.e. holding ABS financed by on-demand deposits, has two dimensions. On the asset side, shocks to the aggregate default rate induces volatility in the income stream. On the liability side, idiosyncratic liquidity shocks introduces further volatility of the stochastic discount factor. The combination of these two sources of risk can be seen in the denominator of (15). Through this channel whenever either liquidity or default risk increases, banks reduce their demand for securities holding. Second, in the last term of the numerator is a dampening effect coming from the general equilibrium effect. Whenever liquidity risk is higher, intermediaries will be asking for a higher spread between the risk free rate and the rate on deposits. This increase in the spread will partially (when the condition of proposition 10 is satisfied) reduce the effects of an increase of liquidity risk in the portfolio decision.
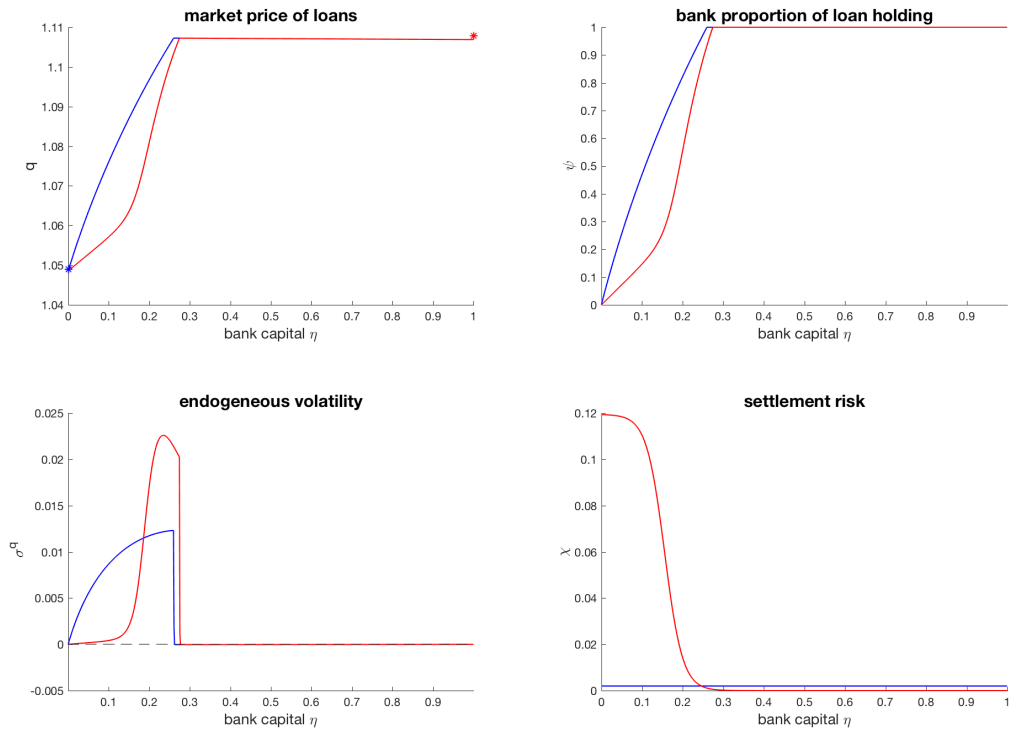
Figure 5: Increased amplification with time varying liquidity risk, the blue lines represents the benchmark close to Brunnermeier and Sannikov (2014) when liquidity risk is constant and almost zero while the red lines features increasing liquidity risk in the crisis region and opens the liability channel of amplification.

Figure 5 provides a comparison between two simulations of the model. The blue lines correspond to the baseline model where liquidity risk is constant and set to a value arbitrarily small for any value of the banking sector capitalization η while the red lines correspond to a model where $\theta(\eta_t)$ has been calibrated to increase at the same time as banks start selling assets to households. This (exogenous) pattern for the liquidity risk is meant to capture the effect of increasing haircuts as endogenous volatility $\sigma_{q,t}$ increases. It can be seen in the lower-right panel. Once reached the crisis regime, where banks start selling assets to households, loan prices

on the secondary market decreases more sharply because this funding liquidity risk increases the overall riskiness of holding leveraged positions (upper-left panel). For the same reasons, banks (fire-)sell ABS at a faster pace (upper-right panel) and endogeneous volatility increases therefore at much higher level (lower-left pannel).

Policy Intervention in a Liquidity Crisis

During liquidity crises, the central bank, being the monopoler of the ultimate mean of settlement is in a prime position to relieve the banking sector and prevent contagion. The central bank can act on the funding side of the liquidity spiral by both injecting reserves in the system (i.e. including to institutions which do not normally have access to the balance sheet of the central bank to work against market fragmentation) and by diminishing aggregate liquidity risk by itself purchasing ABS on the secondary market. The key assumption for this is that central bank does not face liquidity risk when holding assets. For this reason, outright purchases of ABS relieves the financial intermediary of the burden of the liquidity risk associated with the refinancing these assets and, hence, prevents the prices from falling. We first show that, in our model, injecting liquidity stabilizes the economy by shutting down the funding liquidity side of the double spiral. We then show that central bank purchases of ABS has the similar impact as a liquidity injection to reduce liquidity risk. Eventually, we provide some impulse response function for a shock of 15pc on bank capital from steady state with and without policy intervention to illustrate its benefits in sustaining asset prices, loan issuance and output growth. For tractability, we study constant policy rather than state dependent ones.

Proposition 11 (Increasing reserve supplies during a liquidity crisis stabilizes the economy). *Assume that $\theta < 1$ and consider a given equilibrium $\{r(\eta_t), r^d(\eta_t), q(\eta_t), \iota(\eta_t), l(\eta_t), \underline{l}(\eta_t) \hat{c}(\eta_t), \underline{\hat{c}}(\eta_t)\}$ with reserve supply $M_t/n_t = \bar{m}$ then, any equilibrium $\{[r(\eta_t)]^*, [r^d(\eta_t)]^*, [q(\eta_t)]^*, [\iota(\eta_t)]^*, [l(\eta_t)]^*, [\underline{l}(\eta_t)]^* [\hat{c}(\eta_t)]^* [\underline{\hat{c}}(\eta_t)]^*\}$ such that $\bar{m}^* \geq \bar{m}$ has higher asset prices $[q(\eta_t)]^* \geq q(\eta) \forall \eta \in [0,1]$, higher loan issuance $[\iota(\eta_t)]^* \geq \iota(\eta) \forall \eta \in [0,1]$ and therefore higher growth $[y(\eta_t)]^* \geq y(\eta) \forall \eta \in [0,1]$* $\qquad\square$

According to assumption 5, for whatever value of $\theta \in]0, 1[$ liquidity risk will be lower whenever the reserve supply set by the central bank $\bar{m}$ is higher. This lower liquidity risk translates into higher demand from the banking sector for ABS through equation (15) and therefore, in general equilibrium, prices need to be higher to accomodate this higher demand for any value of the state variable η. These higher asset prices then translate into making loan issuance more profitable such that investment and output growth remain at a higher level.

Proposition 12 (Outright security purchases from the central bank during liquidity crises stabilizes the economy). *Assume that Assumption 1 holds and consider a given equilibrium with the net central bank holding of securities being positive ($L^*_{cb,t} > 0$) and a given level central bank liquidity m^*. The exact same equilibrium can be reproduced with zero central bank security holding ($L^{**}_{cb,t} = 0$) and some higher reserve supply $m^{**} > m^*$. Therefore, central bank purchases of ABS works as a substitute to liquidity injection to stabilize the economy.* □

The intuition behind proposition 12 stems from the same underlying principle as Proposition 8, as assumption 1 ensures that the security purchases do not affect the distribution of wealth nor the excess return of their security holdings and because the central bank is not subject to liquidity risk, the only remaining effect is to decrease the aggregate liquidity risk that the banking sector is facing. In order to illustrate the effect of injecting reserves into the economy during a crisis –or, in virtue of proposition 11, asset purchases from the central bank– we simulate numerically the impulse response function of the model to a productivity shock of a drop of 15pc in bank capital from the steady state with and without a liquidity injections that is assumed to be sufficient to completely shut down liquidity risk.

6   Conclusion

In this article, we proposed a path for introducing liquidity risk in a general equilibrium intermediary asset pricing model. With inspirations from the monetary policy
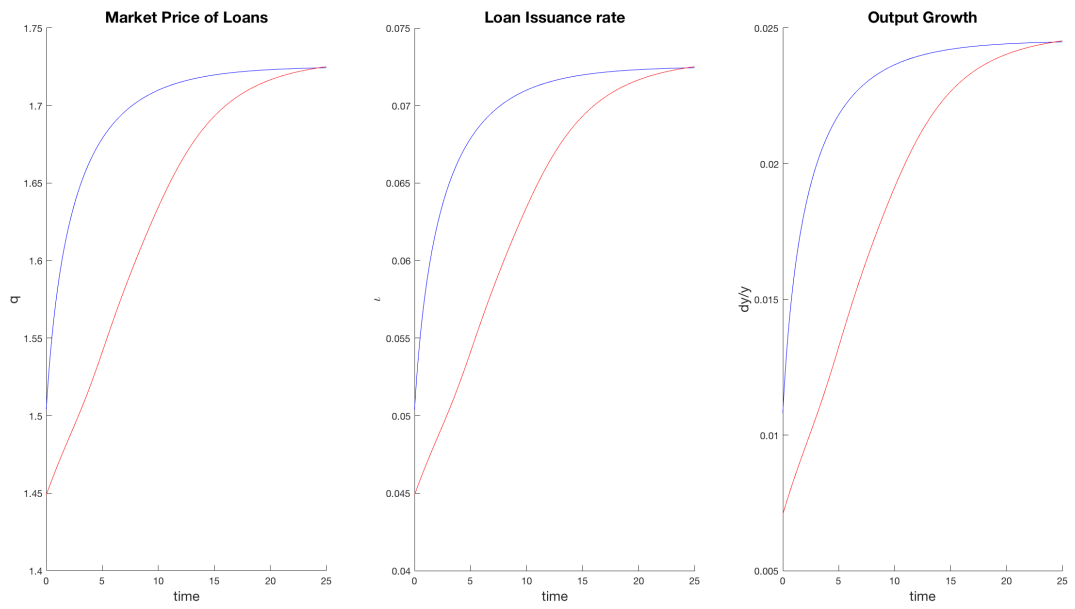
Figure 6: Impulse response function for a 15pc drop of bank capital from steady state without central bank liquidity injection in red and with central bank injection in blue (assuming that this intervention is sufficient to supress any liquidity risk).

implementation and decentralized OTC markets, we did so by assuming that leveraging issuing on demand deposits to hold long capital market assets create extra volatility of wealth: liquidity risk. This setting creates a natural way to introduce a role for central bank reserves as mitigating these frictions. The framework proofs to be powerful in qualitatively mimicking developpments in monetary policy both in normal, pre-2008 times and under liquidity stresses. The model is simple enough to be extended in multiple directions and natural future research includes introducing nominal frictions to understand the interaction between liquidity stresses and macroeconomic stabilization but also to tailor micro-foundations to include market fragmentation when some financial institutions face default risks and don't have access to the central bank balance sheet.

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Appendix

Appendix A: Proofs of Propositions

Deriving FOCs

We first derive the HJB equation for the banking sector. Write the process for bank consumption as:

$$\frac{dc_t}{c_t} = \mu_{c,t}dt + \sigma_{c,t}dZ_t + \tilde{\sigma}_{c,t}d\tilde{Z}_t$$

As the two Brownians are uncorrelated. We can write the HJB as:

$$\rho_t V_t = \log(c_t) + \frac{\partial V_t}{\partial c_t}c_t\mu_{c,t} + \frac{1}{2}\frac{\partial^2 V_t}{\partial c_t \partial c_t}(c_t\sigma_{c,t})^2 + \frac{1}{2}\frac{\partial^2 V_t}{\partial c_t \partial c_t}(c_t\tilde{\sigma}_{c,t})^2$$

Now postulate (guess and verify) that the value function has the following form:

$$V(t, c_t) = V(n_t, \xi_t) = \frac{\log(n_t)}{\rho} + \xi_t$$

where ξ_t follows the process:

$$\frac{d\xi_t}{\xi_t} = \mu_{\xi,t}dt + \sigma_{\xi,t}dZ_t$$

Note that ξ_t is not affected by the idiosyncratic Brownian, as it is idiosyncratic and undiversifiable and therefore only affects the wealth of an individual agent but not the rest of the system. In other words, the *stock of idiosyncratic risk* to which banks are exposed will influence the equilibrium through affecting optimal portfolio holdings but the *realization* of these shocks will be inconsequential. Ito's lemma give us[13]:

$$E_t(dV_t) = V_\xi\mu_{\xi,t}\xi_t + V_n\mu_{n,t}n_t + \frac{1}{2}\left[V_{\xi\xi}\sigma_{\xi,t}^2\xi_t^2 + V_{nn}(\sigma_{n,t}^2 n_t^2 + \tilde{\sigma}_{n,t}^2 n_t^2) + 2V_{\xi n}\sigma_{\xi,t}\xi_t\sigma_{n,t}n_t\right]$$

---

[13]in a system in a recursive equilibrium, state variables characterize the whole system such that V only moves trough time as a deterministic function of other variables. Therefore, $\dot{V} = 0$

Which we can simlify[14] as:

$$\rho_t V(n_t, \rho_t) = \log(\hat{c}_t n_t) + V_n \left(i_t + w_t(\mu_{Rt} + \pi_t - i_t) + w_t^r(i_t^r - i_t) + w_t^d(i_t^d - i_t) - \hat{c}_t - \pi_t \right) n_t$$
$$- V_\xi \mu_{\xi,t} \xi_t + \frac{1}{2} V_{nn} \left[(\sigma_{n,t} n_t)^2 + \left(\chi(w_t^r, \theta_t) w_t^d n \right)^2 \right] + \frac{1}{2} V_{\xi\xi} \left(\xi_t \sigma_{\xi,t} \right)^2 + V_{n\xi} n_t w_t \sigma_{q,t} \xi_t \sigma_{xi,t}$$

Given our guess, we have:

$$V_n = \frac{1}{\rho n_t}; \quad V_{nn} = -\frac{1}{\rho n_t^2}; \quad V_\xi = 1; \quad V_{\xi\xi} = 0; \quad V_{n\xi} = 0$$

HJB therefore simplifies into:

$$\rho \xi_t = \rho \log(\hat{c}_t) + i_t + w_t(\mu_{Rt} + \pi_t - i_t) + w_t^r(i_t^r - i_t) + w_t^d(i_t^d - i_t) - \hat{c}_t - \pi_t$$
$$+ \frac{1}{2} \left(\sigma_{n,t}^2 + (\chi(w_t^r, \theta_t)^2) \right)$$

Applying the *maximum principle*, we take the derivatives of this expression with respect to the control variables $\{\hat{c}_t, w_t, w_t^r, w_t^d\}$ and find the FOCs given in eq. (9), eq. (10), eq. (11) and eq. (12).

Similarly, we can write the HJB equation for households as:

$$\rho \underline{\xi}_t = \rho \log(\underline{\hat{x}}_t) + i_t + \underline{w}_t(\mu_{\underline{R},t} + \pi_t - i_t) + \underline{w}_t^d(i_t^d - i_t) - \underline{\hat{x}}_t - \pi_t + \frac{1}{2} \sigma_{n,t}^2$$

And applying the *maximum principle* gives us Equation (13) and Equation (14).

Proposition 3

From equation 12 and market clearing condition for reserves, we have:

$$i_t = i_t^r + s(\bar{m}_t, \theta_t) \tag{16}$$

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[14]we assume that the max operator is not binding in the domain considered, extending the problem at the limit of the operator does not pose any particular mathematical problem

Combined with the Fischer equation, we have:

$$\pi_t = (i_t^r + s(\bar{m}_t, \theta_t)) - r_t \tag{17}$$

As defined in Definition 3.1, the central bank controls both the nominal interest paid on reserves i_t^r and the supply of reserves $\bar{m}_t$. Equation (17) has therefore one degree of freedom as a given inflation target could be implemented π_t^* through an infinite combination of $i_t^r, \bar{m}_t$ as long as (17) holds. □

Proposition 7

Start from noting that the supply of reserves $\bar{m}_t$ only impacts the economy through equation 12. Now consider the functional form from assumption 5. One can readily see that $\chi(\bar{m}_t, \theta_t) = 0$ when either there is no friction $\theta_t = 0$ or liquidity has reached liquidity satiation threshold $\bar{m}_t > \frac{\tilde{\chi}}{\nu}$. In this case the left hand side of equation 12 is always zero for whatever changes of $\bar{m}_t$ □

Proposition 8

Consider the following transfer rule:

$$\frac{dT_t}{n_t} = w_{cb,t} \left(\mu_{R,t} - r_t^r \right) dt + w_{cb,t} \sigma_{R,t} dZ_t$$

$$\frac{d\underline{T}_t}{\underline{n}_t} = \underline{w}_{cb,t} \left(\mu_{\underline{R},t} - r_t^r \right) dt + \underline{w}_{cb,t} \sigma_{R,t} dZ_t$$

Where $w_{cb,t} = \psi_t L_{cb,t}$ and $\underline{w}_{cb,t} = (1 - \psi_t) L_{cb,t}$. Notice that it satisfies Assumption 1 as we have:

$$dT_t + d\underline{T}_t = \left[\psi_t(\mu_{R,t} - r_t^r) + (1 - \psi_t)(\mu_{\underline{R},t} - r_t^r)\right] L_{cb,t}dt + L_{cb,t}\sigma_{R,t}dZ_t$$
$$= (\mu_{R_{cb},t} - r_t^r) L_{cb,t}dt + L_{cb,t}\sigma_{R_{cb},t}dZ_t$$

Which implies that $dn_{cb,t} = 0$. Therefore, the volalitility of wealth on aggregate risk are given by:

$$\sigma_{n,t} = (w_t\sigma_{R,t} + w_{cb,t}\sigma_{R_{cb},t})\, dZ_t$$
$$\sigma_{\underline{n},t} = (\underline{w}_t\sigma_{\underline{R},t} + \underline{w}_{cb,t}\sigma_{R_{cb},t})\, dZ_t$$

Therefore the FOC conditions of both agents regarding to their security holding can be expressed as :

$$w_t = \frac{\mu_{R,t} - r_t}{\sigma_{R,t}} - w_{cb,t}$$
$$\underline{w}_t\sigma_{\underline{n},t} = \frac{\mu_{\underline{R},t} - r_t^d}{\sigma_{R,t}} - \frac{\rho(1 - \beta)}{\sigma_{R,t}(1 - \underline{w}_t)} - \underline{w}_{cb,t}$$

Both agents exactly offsets any exposure to the position taken by the central bank such that the equilibrium remains unaffected. □

Appendix B: Generalized Model

Appendix A presents a generalized version of the model in two steps. We start from a generalization of the model in Brunnermeier and Sannikov (2014) with E-Z, Heterogeneous Preferences and bank holding loans rather than capital as in (Rochet et al., 2016) as a benchmark model. We the extend this model with liquidity preferences on the household side and with idiosyncratic liquidity risk on the intermediary sector

side.

Brunnermeier and Sannikov (2014) with E-Z, Heterogeneous Preferences and Loans

Preferences Both agents have stochastic differential utility, as developed by Duffie (1992). The utility of agent j over his consumption process c_t^j is defined as

$$U_{jt} = \mathbb{E}_t \left(\int_t^\infty f_j\left(c_{js}, U_{js}\right) ds \right).$$

The function $f_j(c_j, u_j)$ is a normalized aggregator of consumption and continuation value in each period defined as

$$f_j(c, U_j) = \frac{1 - \gamma_j}{1 - 1/\xi_j} U_j \left[\left(\frac{c_j}{((1 - \gamma_j)U_j)^{1/(1-\gamma_j)}} \right)^{1 - 1/\xi_j} - \rho \right]$$

where ρ is the rate of time preference, γ_j is the coefficient of relative risk aversion, and $1/\xi$ is the inter-temporal elasticity substitution (IES) parameter. Each agent choose their optimal consumption c_{jt}, investment risk, and portfolio weights w_{jt} on their capital holdings in order to maximize their discounted infinite life time expected utilities U_{jt}. At any time, the following budget constraint has to be satisfied:

$$\frac{dn_{jt}}{n_{jt}} = \left((1 - w_{jt})i_t + w_{jt}\mu_{R_jt} - \hat{c}_{jt} \right) dt + w_{jt}\sigma_{R_jt}dZ_t.$$

where n_t^j is the wealth of agent j and $\hat{c}_{jt} = c_{jt}/n_{jt}$ his consumption rate, μ_{R_jt} is the real return on loan holdings and i_t is the nominal risk-free rate

Technology The production technology in the economy is given by

$$y_t = r_l l_t.$$

and

$$\frac{dl_{jt}}{l_{jt}} = \left(\Phi \left(\iota_{jt} \right) + \delta - p_{jt} \right) dt + \sigma_t dZ_t$$

The price of an unit of capital is q_t. As the economy only features one stochastic process dZ_t, we can write that the stochastic law of motion of q_t follows:

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t$$

where μ_t^q and σ_t^q are to be determined endogenously in the model using market clearing conditions. We can use Ito's lemma to write the process of the value of capital:

$$\frac{d(q_t l_{jt})}{q_t l_{jt}} = \left(\Phi \left(\iota_{jt} \right) + \delta - p_{jt} + \mu_t^q + \sigma_t \sigma_t^q \right) dt + \left(\sigma_t + \sigma_t^q \right) dZ_t.$$

Hence, the return on physical asset is:

$$dR_{jt} = \underbrace{\left(\frac{r_l - \iota_{jt}}{q_t} + \Phi \left(\iota_{jt} \right) + \delta - p_{jt} + \mu_t^q + \sigma_t \sigma_t^q \right)}_{\mu_{R_j t}} dt + \underbrace{\left(\sigma_t + \sigma_t^q \right)}_{\sigma_{R_j t}} dZ_t.$$

Solve the HJB We will guess and verify that the homotheticity of preferences allows us to write the value function for agents of type j as:

$$U_{jt} = V_j \left(n_{jt}, \eta_t \right) = \frac{\left(n_{jt} \right)^{1-\gamma_j}}{1 - \gamma_j} \frac{v_j(\eta_t)}{\left(\eta_t q_t \right)^{1-\gamma_j}} = \frac{\left(n_{jt} \right)^{1-\gamma_j} \widetilde{v}_j(\eta_t)}{1 - \gamma_j}, \tag{18}$$

where the state variable η_t is defined by

$$\eta_t = \frac{n_t^i}{n_t^i + n_t^h}$$

and follows

$$\frac{d\eta_t}{\eta_t} = \mu_t^\eta dt + \sigma_t^\eta dZ_t.$$

Thus, we can write the Hamilton Jacobi Bellman equation corresponding to the problem as

$$0 = \max_{\hat{c}_{jt},\sigma_t,\iota_{jt},w_{jt}} f^j\left(\hat{c}_{jt}n_{jt}, U_{jt}\right) \qquad (19)$$

$$+ \left((1-w_{jt})r_t + w_{jt}\mu_{R_j t} - \hat{c}_{jt}\right) n_{jt} V_n^j(n_{jt},\eta_t) + \frac{1}{2}\left(w_{jt}\sigma_{Rt}n_{jt}\right)^2 V_{nn}^j(n_{jt},\eta_t)$$

$$+ \mu_t^\eta \eta_t V_\eta^j(n_{jt},\eta_t) + \frac{1}{2}\left(\sigma_t^\eta \eta_t\right)^2 V_{\eta\eta}^j(n_{jt},\eta_t)$$

$$+ \sigma_t^\eta \eta_t w_{jt}\sigma_{Rt}n_{jt} V_{\eta n}^j(n_{jt},\eta_t).$$

From equation (18), we have

$$f_j\left(\hat{c}_{jt}n_{jt}, V^j(n_{jt},\eta_t)\right) = \frac{(n_{jt})^{1-\gamma_j}\,\widetilde{v}^j(\eta_t)}{1-1/\xi}\left[\left(\frac{\hat{c}_{jt}}{(\widetilde{v}^j(\eta_t))^{\frac{1}{1-\gamma_j}}}\right)^{1-1/\xi} - \rho\right],$$

$$V_n^j(n_{jt},\eta_t) = (n_{jt})^{-\gamma_j}\,\widetilde{v}^j(\eta_t),$$

$$V_{nn}^j(n_{jt},\eta_t) = -\gamma_j(n_{jt})^{-\gamma_j-1}\,\widetilde{v}^j(\eta_t),$$

Thus, we can rewrite (19) as

$$0 = \max_{\hat{c}_{jt},\sigma_t,\iota_{jt},w_{jt}} \frac{1}{1-1/\xi}\left[\left(\frac{\hat{c}_{jt}}{(\widetilde{v}^j(\eta_t))^{\frac{1}{1-\gamma_j}}}\right)^{1-1/\xi} - \rho\right] + (1-w_{jt})r_t + w_{jt}\mu_{R_j t} - \hat{c}_{jt} - \frac{\gamma}{2}\left(w_{jt}\sigma_{Rt}\right)^2$$

$$+ \frac{\mu_t^\eta \eta_t}{1-\gamma}\frac{\widetilde{v}_\eta^j(\eta_t)}{\widetilde{v}^j(\eta_t)} + \frac{1}{2}\frac{(\sigma_t^\eta \eta_t)^2}{1-\gamma}\frac{\widetilde{v}_{\eta\eta}^j(\eta_t)}{\widetilde{v}^j(\eta_t)} + \sigma_t^\eta \eta_t w_{jt}\sigma_{Rt}\frac{\widetilde{v}_\eta^j(\eta_t)}{\widetilde{v}^j(\eta_t)}.$$

which confirms that $v^j(\eta_t)$ only depends on η_t and not on n_{jt}.

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Optimality Conditions   We can take the first order conditions:

$$(\hat{c}_{jt})^{-1/\xi} = \left(\widetilde{v}^j(\eta_t)\right)^{\frac{1-1/\xi}{1-\gamma}}$$

$$\hat{c}_{jt} = \left(\widetilde{v}^j(\eta_t)\right)^{\frac{1-\xi}{1-\gamma}}$$

$$\mu_{R_jt} - r_t - \gamma_j w_{jt} \left(\sigma_{Rt}\right)^2 + \sigma_t^\eta \eta_t \sigma_{Rt} \frac{\widetilde{v}_\eta^j(\eta_t)}{\widetilde{v}^j(\eta_t)} = 0$$

$$1/q_t = \Phi_\iota(\iota_{jt})$$

Note an useful expression for the Sharpe ratio:

$$\varsigma_{jt} \equiv \frac{\mu_{R_jt} - r_t}{\sigma_{Rt}} = \gamma_j w_{jt} \sigma_{Rt} - \frac{\widetilde{v}_\eta^j(\eta_t)}{\widetilde{v}^j(\eta_t)} \sigma_t^\eta \eta_t$$

Plugging in the optimality conditions gives:

$$0 = \frac{1}{1-1/\xi} \left[\left(\widetilde{v}^j(\eta_t)\right)^{\frac{1-\xi_j}{1-\gamma_j}} - \rho \right] + r_t - \hat{c}_{jt} + \frac{\gamma_j}{2} \left(w_{jt}\sigma_{Rt}\right)^2$$
$$+ \frac{\mu_t^\eta \eta_t}{1-\gamma} \frac{\widetilde{v}_\eta^j(\eta_t)}{\widetilde{v}^j(\eta_t)} + \frac{1}{2} \frac{(\sigma_t^\eta \eta_t)^2}{1-\gamma} \frac{\widetilde{v}_{\eta\eta}^j(\eta_t)}{\widetilde{v}^j(\eta_t)}.$$

Prices as Functions of $v^j(\eta_t)$ The law of motion of n_{jt} is given by:

$$\frac{dn_{jt}}{n_{jt}} = r_t dt + w_{jt}(\sigma_t + \sigma_t^q)\left(\varsigma_{jt}dt + dZ_t\right) - \hat{c}_{jt}dt$$

The law of motion of $q_t k_t$ is given by:

$$\frac{d(q_t k_t)}{q_t k_t} = \left(\Phi(\iota_t) + \mu_t^q + \sigma_t \sigma_t^q\right)dt + (\sigma_t + \sigma_t^q)\, dZ_t$$

$$\psi_t \equiv \frac{w_t^i n_t^i}{w_t^i n_t^i + w_t^h n_t^h} = w_t^i \eta_t$$

We can therefore use Ito's lemma to write the law of motion of η_t as:

$$\frac{d\eta_t}{\eta_t} = \left(r_t + w_t^i(\sigma_t^i + \sigma_t^q)\varsigma_t^i - \hat{c}_t^i - \Phi(\iota_t) - \mu_t^q - \sigma_t \sigma_t^q + (\sigma_t + \sigma_t^q)^2 - w_t^i(\sigma_t^i + \sigma_t^q)(\sigma_t + \sigma_t^q) \right) dt$$

$$+ (w_t^i(\sigma_t^i + \sigma_t^q) - \sigma_t + \sigma_t^q) dZ_t$$

$$= \left(r_t + w_t^i(\sigma_t^i + \sigma_t^q)(\varsigma_t^i - \sigma_t - \sigma_t^q) - \hat{c}_t^i - \Phi(\iota_t) - \mu_t^q - \sigma_t \sigma_t^q + (\sigma_t + \sigma_t^q)^2 \right) dt$$

$$+ (w_t^i(\sigma_t^i + \sigma_t^q) - \sigma_t + \sigma_t^q) dZ_t$$

We can use the market clearing condition for consumption find q_t.

$$\left(\hat{c}_t^i \eta_t + \hat{c}_t^h(1 - \eta_t) \right) q_t = \psi_t(a^i - \iota_t^i) + (1 - \psi_t)(a^h - \iota_t^h)$$

We can use the market clearing condition for capital find r_t.

$$w_t^i \eta_t + w_t^h(1 - \eta_t) = 1.$$

We need q_t' and q_t'' to find σ_t^q and μ_t^q:

$$q_t \sigma_t^q = q_t' \sigma_t^\eta \eta_t,$$

$$q_t \mu_t^q = q_t' \mu_t^\eta \eta_t + \frac{1}{2} q_t'' \left(\sigma_t^\eta \eta_t \right)^2.$$

Because the continuation value $V^j(n_{jt}, \eta_t)$ is a martingale, we have that:

$$\rho V^j(n_{jt}, \eta_t) = \frac{(\hat{c}_{jt} n_{jt})^{1-\gamma_j}}{1 - \gamma_j} + \mu_t^{V,j} V^j(n_{jt}, \eta_t).$$

Since

$$\frac{dk_t^{1-\gamma_j}}{k_t^{1-\gamma_j}} = (1-\gamma_j)\left(\Phi(\iota_t) - \frac{\gamma_j(1-\gamma_j)}{2}\sigma^2\right)dt + (1-\gamma_j)\sigma dZ_t,$$

and

$$V^j(n_{jt}, \eta_t) = \frac{v^j(\eta_t)(n_{jt})_t^{1-\gamma_j}}{(1-\gamma_j)(q_t\eta_t)^{1-\gamma_j}} = \frac{v^j(\eta_t)k_t^{1-\gamma_j}}{1-\gamma_j},$$

we get

$$0 = \frac{1-\gamma_j}{1-1/\xi_j}\left[\left(\frac{v^j(\eta_t)}{(q_t\eta_t)^{1-\gamma_j}}\right)^{\frac{1-\xi_j}{1-\gamma_j}} - \rho_j\right] + \mu_t^{v,j} + (1-\gamma_j)\left(\Phi(\iota_t) - \frac{\gamma_j}{2}\sigma^2\right) + (1-\gamma_j)\sigma\sigma_t^{v,j}.$$

We obtain σ_t^v from

$$v^j(\eta_t)\sigma_t^{v,j} = v_\eta^j(\eta_t)\sigma_t^\eta\eta_t.$$

We can then apply the method of finite difference to

$$v^j(\eta_t)\mu_t^{v,j} = v_\eta^j(\eta_t)\mu_t^\eta\eta_t + \frac{1}{2}v_{\eta\eta}^j(\eta_t)\left(\sigma_t^\eta\eta_t\right)^2 + v_{jt}(\eta_t).$$

Boundary Conditions   The boundaries at η close to 0 or 1 can be found by solving the following system of equation:

$$0 = \frac{1}{1-1/\xi_j}\left[\left(\frac{v^j}{(q\eta)^{1-\gamma_j}}\right)^{\frac{1-\xi_j}{1-\gamma_j}} - \rho_j\right] + (1-\gamma_j)\left(\Phi - \frac{\gamma_j}{2}\sigma^2\right)$$

$$\left(\frac{v^i}{(q\eta)^{1-\gamma_i}}\right)^{\frac{1-\xi_i}{1-\gamma_i}} + \left(\frac{v^h}{(q(1-\eta))^{1-\gamma_h}}\right)^{\frac{1-\xi_h}{1-\gamma_h}} = \psi(a^i - \iota^i) + (1-\psi)(a^h - \iota^h),$$

where we assumed that $v^j(\eta) = v^j$.

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Solving the Complete Model with Liquidity Risk and Deposit in the Utility

We add idiosyncratic risk and central bank reserves on banker's problem and deposit in the utility on household's problem. We furthermore assume that contracts are written in nominal terms but that inflation is deterministic.

Banker Problem   Removing, individual indices, we write the law of motion of wealth as:

$$\frac{dn_t}{n_t} = (r_t + w_t(\mu_{Rt} - i_t) + w_t^r(i_t^r - i_t) - \hat{c}_t)\, dt + w_t\sigma_{Rt}dZ_t + \chi(w_t^r, \theta_t)\min(w_t - 1)d\tilde{Z}_t$$

Let's write:

Assuming that $w_t > 1$ which will be true in equilibrium, write the HJB equation:

$$0 = \max_{\hat{c}_t, \iota_t, w_t} \frac{1}{1 - 1/\xi} \left[\left(\frac{\hat{c}_t}{(\widetilde{v}(\eta_t))^{\frac{1}{1-\gamma_j}}} \right)^{1-1/\xi} - \rho \right] + r_t + w_t(\mu_{Rt} - i_t) + w_t^r(i_t^r - i_t) - \hat{c}_t$$

$$-\frac{\gamma}{2}(w_t\sigma_{Rt})^2 - \frac{\gamma}{2}(\chi(w_t^r, \theta_t)(w_t - 1))^2 + \frac{\mu_t^\eta \eta_t}{1 - \gamma}\frac{\widetilde{v}_\eta(\eta_t)}{\widetilde{v}(\eta_t)} + \frac{1}{2}\frac{(\sigma_t^\eta \eta_t)^2}{1 - \gamma}\frac{\widetilde{v}_{\eta\eta}(\eta_t)}{\widetilde{v}(\eta_t)}$$

$$+\sigma_t^\eta \eta_t (w_t\sigma_{Rt} + \chi(w_t^r, \theta_t)(w_t - 1))\frac{\widetilde{v}_\eta(\eta_t)}{\widetilde{v}(\eta_t)}$$

Optimality Conditions   Consumption:

$$\hat{c}_t = (\widetilde{v}(\eta_t))^{\frac{1-\xi}{1-\gamma}}$$

Risky Asset:

$$\mu_{Rt} - i_t - \gamma w_t \left(\sigma_{Rt}\right)^2 + \gamma \chi(w^r, \theta)^2 - \gamma w_t \chi(w^r, \theta)^2 + \sigma_t^\eta \eta_t (\sigma_{Rt} + \chi(w^r, \theta)) \frac{\widetilde{v}_\eta(\eta_t)}{\widetilde{v}(\eta_t)} = 0$$

$$w_t = \frac{1}{\gamma} \left[\frac{\mu_t - i_t}{(\sigma_{Rt}^2 + \chi_t^2)} + \sigma_t^\eta \eta_t \frac{\widetilde{v}_{t\eta}(\eta_t)}{\widetilde{v}(\eta_t)} (\sigma_{Rt} + \chi_t) \right] + \frac{\chi_t^2}{(\sigma_{Rt}^2 + \chi_t^2)}$$

Reserves:

$$(i_t^r - i_t) - \gamma(w_t - 1)^2 \chi(w^r, \theta) \chi_{w^r}(w_t^r, \theta) + \sigma_t^\eta \eta_t \frac{\widetilde{v}_\eta(\eta_t)}{\widetilde{v}(\eta_t)} (w_t - 1) \chi_{w^r}(w^r, \theta) = 0$$

$$(i_t^r - i_t) = \gamma(w_t - 1)^2 \chi(w_t^r, \theta) \chi_{w^r}(w_t^r, \theta) - \sigma_t^\eta \eta_t \frac{\widetilde{v}_\eta(\eta_t)}{\widetilde{v}(\eta_t)} (w_t - 1) \chi_{w^r}(w_t^r, \theta) \equiv -s(w_t^r, \theta_t)$$

Investment:

$$1/q_t = \Phi_\iota(\iota_t, \sigma_t)$$

Note a useful expression for the Sharpe ratio:

$$\varsigma_t \equiv \frac{\mu_{Rt} - r_t}{\sigma_{Rt}} = \left[\pi_t + \gamma w_t(\sigma_{Rt}^2 + \chi_t) - \gamma \chi_t - \sigma^\eta \eta(\sigma_{Rt} + \chi_t) \frac{\widetilde{v}_\eta(\eta_t)}{\widetilde{v}(\eta_t)} \right] / \sigma_{Rt}$$

To go further assume the following functional form for χ:

$$\chi(w_t^r, \theta_t) = \max \left\{ (1 - \theta_t)(\bar\chi - \nu w_t^r), 0 \right\}$$

where $w^r = \frac{\bar\chi}{\nu}$ is the point of liquidity satiation for banks after which banks do not reduce exposure to liquidity risk anymore from holding more reserves. When satiation is not binding, we have:

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$$\chi_{w^r}(w_t^r, \theta_t) = -(1-\theta)\nu \qquad (20)$$

Monetary policy is implemented picking both the quantity w^r and rate i^r. Whenever the central bank only care about inflation stabilization, these two tools give one degree of freedom. Inflation is pinned down through the Fischer equation:

$$i_t = r_t + \pi_t$$

Which we inject in the asset pricing condition for the reserves to find:

$$r_t + \pi_t - i_t^r = s(w_t^r, \theta_t)$$

rearranging:

$$\pi_t = \left(s(w_t^r, \theta_t) + i_t^r\right) - r_t$$

We then on assume that inflation is stabilized using interest on reserves i_t^r and that the central bank manipulates the liquidity spread $s(w_t^r, \theta_t)$ by affecting the aggregate supply of reserves in the economy w_t^r according to money market tightness θ_t which also affects the amount of liquidity risk in the model χ_t and therefore excess cost of leverage which banks passe on asset prices.

Adding Deposit-in-the-Utility of Households

Households   Households have preference for liquid deposits in the utility function: They maximize their EZ utility function:

$$V_t^i = \max_{\{w_t, w_t^d, \hat{c}_t\}_t^\infty} \left[\int_0^\infty f^i(x_\tau^i, V_\tau^i) d\tau \right]$$

where X_t is a Cobb-Douglas composite of consumption and liquidity services of deposits:

$$x(c, m) = c^\beta d^{1-\beta}$$

Which we can rewrite as:

$$x(c, m) = \hat{x}n = \hat{c}^\beta \left(w^d\right)^{1-\beta} n$$

Their law of motion for deposit is given by:

$$\frac{dn_t}{n_t} = \left(r_t + w_t(\mu_{R,t} - r_t) + w_t^d(r_t^d - r_t) - w^h r_t - \frac{c_t}{n_t}\right) dt + w_t \sigma_{R,t} dZ_t$$

FOC conditions are given by:

$$f_{\hat{c}}(\hat{c}_t, w_t^d, w_t^h, n_t, V_t) = V_n(n_t, \eta_t)$$

$$(\mu_{R,t} - r_t)n_t V_n(n_t, \eta_t) + (w_t(\sigma_{R,t})^2 n_t^2 V_{nn}(n_t, \eta_t) + \sigma_{\eta,t}\sigma_{R,t}n_t\eta_t V_{\eta n}(n_t, \eta_t) = 0 \quad (21)$$

$$0 = f_{w^d}(\hat{c}_t, w_t^d, w_t^h, n_t, V_t) + (r_t^d - r_t)n_t V_n(n_t, \eta_t) \quad (22)$$

where :

$$f_{\hat{c}} = (1 - \gamma)V_t \left[\frac{(w_t^d)^{1-\beta}n_t}{[(1-\gamma)V_t]^{\frac{1}{1-\gamma}}}\right]^{1-\frac{1}{\xi}} \beta\hat{c}_t^{\beta-1-\frac{\beta}{\xi}}$$

Using (1) gives:

$$f_{\hat{c}} = \beta n_t^{(1-\gamma)}\tilde{v}^{\frac{1}{\xi}}\left(w_t^d\right)^{(1-\beta)(1-\frac{1}{\xi})}\hat{c}_t^{\beta-1-\frac{\beta}{\xi}}$$

and

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$$f_{w^d} = (1-\gamma)V_t \left[\frac{\hat{c}_t^\beta n_t}{[(1-\gamma)V_t]^{\frac{1}{1-\gamma}}} \right]^{1-\frac{1}{\xi}} (1-\beta)\left(w_t^d\right)^{-\beta-\frac{1-\beta}{\xi}}$$

Using (1) gives:

$$f_{w^d} = (1-\beta)n^{(1-\gamma)}\tilde{v}^{\frac{1}{\xi}}\hat{c}_t^{\beta(1-\frac{1}{\xi})}\left(w_t^d\right)^{-\beta-\frac{1-\beta}{\xi}}$$

Note that we can rewrite the spread between the money market rate and deposit as:

$$(r_t - r_t^d) = \tilde{v}^{\frac{1}{\xi}}\hat{c}_t^{\beta(1-\frac{1}{\xi})}\left(w_t^d\right)^{-\beta-\frac{1-\beta}{\xi}}(1-\beta)$$

Whenever there is no preference for the liquidity, the spread between the rate on deposits and the money market rate is zero while consumption rate goes back to the baseline model.

Alternative Problem: No Non-Deposit Risk-Free Debt Assume that a liability of the bank is necessarily a deposit:

$$w_t + w_t^d = 1 \tag{23}$$

Which transforms the law of motion of n_t as :

$$\frac{dn_t}{n_t} = \left(r_t^d + w_t(\mu_{R,t} - r_t^d) - \frac{c_t}{n_t}\right)dt + w_t\sigma_{R,t}dZ_t$$

giving the first order conditions:

$$f_{\hat{c}}(\hat{c}_t, w_t^d, n_t, V_t) = V_n(n_t, \eta_t)$$

$$-f_{w^d}(\hat{c}_t, w_t^d, n_t, V_t) + (\mu_{R,t} - r_t^d)n_t V_n(n_t, \eta_t) + (w_t(\sigma_{R,t})^2 n_t^2 V_{nn}(n_t, \eta_t)$$
$$+ \sigma_{\eta,t}\sigma_{R,t}n_t\eta_t V_{\eta n}(n_t, \eta_t) = 0$$

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$$\frac{-f_{w^d}}{\tilde{v}(\eta_t)n^{1-\gamma}} + (\mu_{Rt} - r_t^d) - w_t\sigma_{Rt}^2 + \sigma_{\eta t}\sigma_{Rt}\eta_t\frac{\tilde{v}'(\eta_t)}{\tilde{v}(\eta_t)} = 0$$

$$\varsigma_t \equiv \frac{(\mu_{Rt} - r_t^d)}{\sigma_{Rt}} = w_t\sigma_{Rt} - \sigma_{\eta t}\eta_t\frac{\tilde{v}'(\eta_t)}{\tilde{v}(\eta_t)} + \frac{f_{w^d}}{\tilde{v}(\eta_t)n^{1-\gamma}}$$

$$\varsigma_t \equiv \frac{(\mu_{Rt} - r_t^d)}{\sigma_{Rt}} = w_t\sigma_{Rt} - \sigma_{\eta t}\eta_t\frac{\tilde{v}'(\eta_t)}{\tilde{v}(\eta_t)} + \frac{(1-\beta)\tilde{v}^{\frac{1}{\xi}}\hat{c}_t^{\beta(1-\frac{1}{\xi})}\left(w_t^d\right)^{-\beta-\frac{1-\beta}{\xi}}}{\tilde{v}(\eta_t)}$$