Optimal Timing of Calling-in Large-Denomination Banknotes: Option Value and Strategic Interactions

Thomas Link*

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E A R L Y   D R A F T

Abstract

Rogoff has proposed to phase out paper currency to make economies cashless step by step. Such attempts that would start out with the calling-in of large-denomination banknotes could mitigate several social and economic problems like those posed by the effective lower bound on monetary policy rates. This paper provides theoretical evidence that the welfare gains from international coordination of calling-in moves could be substantial. We solve a stochastic stopping time game where two central banks choose exercise strategies for their 'options' to call in large banknotes, and where decisions must be made under uncertainty about the future path of the natural rate of interest. The analysis shows, firstly, that a 'wait-and-see'-value of the option to call in large notes will induce central banks to delay any calling-in moves, and secondly, that policy coordination will preserve this option value and will avoid welfare losses that result from suboptimal timing decisions.

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Keywords: cashless economy, phase-out of paper currency, international monetary policy coordination, option value, stopping time game

*Heinrich Heine University Düsseldorf Department of Economics, Universitätsstraße 1, 40225 Düsseldorf, Germany, phone: +49/(0)211/81 15318, email: thomas.link@hhu.de. The author is grateful to seminar participants in Düsseldorf, and especially to Ulrike Neyer for the discussions and many helpful comments.

1 Introduction

Rogoff (2017) has proposed to phase out paper currency to make economies cashless step by step. Our paper proceeds on the assumption that a rational decision maker in power over the legal tender must at least consider starting such a process – which then would begin, like in Rogoff's proposal, with the calling-in of large-denomination banknotes.[1] While most of the debate so far has been centered around the private and social costs and benefits of such an attempt, we introduce the issue of optimal timing into the discussion and evaluate the effects of economic uncertainty and strategic interactions between countries in this respect. The analysis of a timing game shows the necessity of international policy coordination in a strategic and uncertain environment in order to preserve national flexibility and prevent socially inefficient timing decisions.

Clearly, although their objective functions might differ to a great extent, in the end, any appropriate action will be the decision of a fiscal or of a monetary authority based on an analysis of social, political, or economic costs and benefits. The loss of seignorage revenues, symbolic value, diversity of transaction media, or of civil liberties could make it costly to call-in large-denomination banknotes. However, there are two main arguments against their issuance. The first is a relatively trivial one: their usage in the shadow and underground economy. The second argument is more complex and certainly has been re-evaluated during the last decade in the light of the low interest rate environment that prevailed in America and Europe: from a monetary policy point of view paper currency is a friction that induces the constraint of an effective lower bound on policy rates.[2] Of course, monetary policy at the effective lower bound (ELB) is not powerless and can draw on unconventional instruments such as forward guidance or quantitative easing. But the question is simply whether the world would be better off if this friction were eliminated and the ELB-constraint thus relaxed.

[1] In most of our discussion we will use the term *"calling-in"* in a broad definition. This ranges, on the one side, from removing certain banknotes' status as legal tender without delay (similar to the calling-in of 500- and 1000-rupee notes in India in 2016) to, on the other side, a 'soft' version where certain banknotes are removed from circulation only gradually over time by simply stopping their issuance (which is the ECB's approach of phasing out the 500-euro note). In Section 2 we make a technical assumption and define a *"calling-in"* move as a one-shot policy that stops the issuance of banknotes of certain denominations with immediate effect.

[2] See Rogoff (1998), Rogoff (2015), and Rogoff (2017) for a detailed discussion of all of these points.

Whatever the answer, potential welfare gains from relaxing the ELB-constraint certainly depend on the likeliness of ELB-episodes in the future. Some commentators have argued that for the advanced economies this likeliness is much higher now than it was in past decades – a circumstance that they attribute to persistently low inflation rates and above all to a decline in real interest rates.[3] Whatever the reasons of this decline in real rates, Holston, Laubach, and Williams (2017) estimate that the natural (or equilibrium) real rates of interest in Canada, the euro area, the United Kingdom, and the U.S. have fallen considerably over the past 25 years, especially in the aftermath of the financial crisis (in the U.S. close to zero, in the euro area even below zero). Kiley and Roberts (2017) assess potential consequences of such a fall and estimate in a simulation study based on a DSGE and the large-scale econometric FRB/US model that the U.S. economy could be stuck at the ELB up to 40% of the time and then with an average duration of more than two years if a steady state level of the natural real rate of interest of 1% would prevail while the Fed targets a 2% inflation level.[4] So, although there is much uncertainty in this respect, considering the negative output and inflation gaps that are associated with ELB-episodes, the welfare gains from relaxing the ELB-constraint could be non-negligible.

Now, the idea is to reduce the ELB by removing large-denomination banknotes from circulation. As large notes are relatively easy to store and to transport they facilitate a flight into cash should monetary policy rates ever be set significantly below zero. So, the smaller the largest denominations available, the higher the hurdles to take a flight into cash, and thus the lower the ELB on policy rates. In this respect, a benevolent central banker with a clear mandate for price stability and/or full employment should face a far more well-defined decision problem than governments do (which might place more weight on various political and libertarian arguments or on the preservation of symbolic value that large banknotes have, and which might place far less weight on any ELB-issues). This is where our analysis starts.

We formalize the decision problem of a central banker who has the option to call in large denomination banknotes. The exercise of this option on the one hand entails the commitment to future losses, for instance, of credibility or of seignorage revenues, but on

[3]See Rogoff (2017), Kiley and Roberts (2017), or Holston, Laubach, and Williams (2017).

[4]See Kiley and Roberts (2017, p. 22).

the other hand leads to welfare gains from relaxing the ELB-constraint. The crucial point is that these gains are state-dependent and a function of the natural rate of interest, whose future path is stochastic. An essential feature of such a situation – wich is a key argument from the *'Real Options'* literature – is that decision makers with the option to time their actions freely but under uncertainty about the future sometimes should choose a *'wait-and-see'* policy until the expected net benefits of making a move are far greater than zero.[5] So, even if the expected welfare gains from calling-in large banknotes today exceeded any expected deadweight loss, there could be a rational incentive to postpone such a calling-in move to a future date. Our analysis shows how the optimal timing of calling-in large-denomination banknotes depends on the properties of the stochastic process that governs the natural rate of interest.[6] But, on the contrary, the timing decision in a strategic environment can be quite different if any advantage to a first- or second-mover leads to preemptive or free-riding behavior.[7] So, in the second stage of our analysis, we formalize and discuss the coordination problem that major central banks will face if there are externalities to changes in their own currency's denominational structure (as noted by Rogoff (2015), a late or never moving country could benefit from an an increase in its seignorage revenues due to currency substitution after a first-mover has phased-out its large banknotes). A non-cooperative stochastic two-player stopping-time game shows how strategic considerations can lead to inefficient timing decisions and how these inefficiencies further depend on the stochastic properties of the natural rate of interest. The welfare gains from international policy coordination in this respect rationalize that major central banks should start out early enough with the coordination of their actions.

Section 2 considers a central bank's decision problem in a one-country setting. In Section 2.1 we set up an optimal stopping model of calling-in large-denomination banknotes. Section 2.2 analyzes the optimal stopping problem in the deterministic and Section 2.3 in the stochastic case. Section 3 considers the stopping time game between two central banks in a two-country setting. The game is formalized in Section 3.1 and solved in Sec-

[5]See, for instance, Dixit and Pindyck (1994).

[6]In the first stage of our analysis we focus on the mean, the volatility, and the speed of mean-reversion of an Ornstein-Uhlenbeck process.

[7]For a discussion of strategic interactions (although in an industrial organization context) see, for instance, Fudenberg and Tirole (1985) or Thijssen, Huisman, and Kort (2012) for preemption and Weeds (2002) for strategic delay in timing games.

tion 3.2 in a deterministic and in Section 3.3 in a stochastic setting. Section 4 contains some concluding remarks.

2 One-Country Setting

2.1 An Optimal Stopping Model of Calling-In Large-Denomination Banknotes

A central bank has the option to call in large-denomination banknotes. The calling-in move, i.e. option exercise, is a one-shot policy that stops the issuance of banknotes of the n largest denominations with immediate effect ($n \in \mathbb{N}_{>0}$). The reputational costs of reversing this kind of change in the denomination structure are prohibitively high, thus the change is irreversible. The timing of the calling-in move is chosen by the central bank at its own discretion. Time is continuous and the time horizon is $[0, \infty)$.

The basis in the central bank's decision-making process is the utility function U_t that states flow utility in period $t \in [0, \infty)$. U_t is the welfare measure for the effects of the calling-in move. Period utility is zero until the option is exercised. Once that has been done, a deterministic flow utility $\omega \in \mathbb{R}_{>0}$ enters U_t with negative sign, a stochastic flow utility G_t with positive sign, such that $U_t = G_t - \omega$. The known constant ω states the welfare loss, the function G_t the welfare gains resulting from the exercise of the calling-in option. It is natural to assume that the period gain G_t is larger, the lower the natural rate of interest, r_t, is in period t, such that the gain function can be defined as

$$G(r_t) = g - r_t, \tag{1}$$

with $g \in \mathbb{R}$ being a constant gain parameter. The natural rate of interest, r_t, follows an Ornstein-Uhlenbeck process with dynamics

$$dr_t = \theta(r^{ss} - r_t)dt + \sigma dB_t, \tag{2}$$

where B_t is Brownian motion and $\theta \in \mathbb{R}_{>0}$, $r^{ss} \in \mathbb{R}$, and $\sigma \in \mathbb{R}_{\geq 0}$ are known constants. The rate at which the future is discounted is $\delta \in \mathbb{R}_{>0}$.

The central bank will exercise the calling-in option the first time r_t hits or comes below the critical level $\underline{r}$. This *stopping rule* is the solution of the optimal stopping problem which also determines the function $V(r)$ that states the value of the calling-in option in terms of cumulative utility. $\underline{r}$ and $V(\cdot)$ are to be determined.

2.2 Optimal Timing in a Deterministic Setting

The path of the natural rate of interest under perfect foresight with $\sigma = 0$ is given by a deterministic function of the instantaneous rate r_0 and time t with

$$r_t = r_0 \cdot \exp(-\theta t) + r^{ss} \cdot (1 - \exp(-\theta t)) \,. \tag{3}$$

The present discounted value (at $t = 0$) of cumulative future utility *given* the calling-in move is made in period $T \geq 0$ thus is a deterministic function F of T and the instantaneous natural rate of interest r_0 with

$$F(r_0, T) = \int_T^{\infty} (G(r_t) - \omega) \cdot \exp(-\delta t) dt \tag{4}$$

$$= \frac{1}{\delta} \cdot (g - r^{ss} - \omega) \cdot \exp(-\delta T) \;-\; \frac{1}{\delta + \theta} \cdot (r_0 - r^{ss}) \cdot \exp(-(\delta + \theta)T). \tag{5}$$

Maximizing F over $T \in [0, \infty)$ yields the optimal timing, T^*, of the calling-in move. The value function V thus is a deterministic function of r_0 with

$$V(r_0) := \max_T F(r_0, T) = F(r_0, T^*). \tag{6}$$

The next two scenarios provide some intuition on different regions of the parameter space at $\sigma = 0$ where immediate/postponed/no exercise is optimal.

Scenario I (Static Case) For $r_0 = r^{ss}$, the function (4) simplifies to $F(r_0, T) = \frac{1}{\delta} \cdot (g - r^{ss} - \omega) \cdot \exp(-\delta T)$ and monotonicity implies $T^* \in \{0\} \cup \{\infty\}$ for $g - \omega \neq r^{ss}$. For $r_0 = g - \omega$ the central bank is just indifferent between any $T^* \in [0, \infty)$. The option is exercised never if $r_0 > g - \omega$, and immediately if $r_0 < g - \omega$. Thus, the critical level $\underline{r} = g - \omega$.

Scenario II (Convergence to a Low Steady State) If the natural rate of interest converges monotonically to a low steady state with $r_0 > g - \omega > r^{ss}$ optimality requires the central bank to postpone the calling-in move to some finite $T^* > 0$. The first-order condition for an interior maximum of (4) is

$$(g - r^{ss} - \omega) \cdot \exp(-\delta T) = (r_0 - r^{ss}) \cdot \exp(-(\delta + \theta)T) \tag{7}$$

and yields

$$T^* = \frac{1}{\theta} \cdot \ln \left(\frac{r_0 - r^{ss}}{g - r^{ss} - \omega} \right). \tag{8}$$

So, the option is not exercised until the natural rate of interest hits the critical level $\underline{r} = g - \omega$. It is optimal to postpone the calling-in move to some $T^* > 0$ even if the present discounted value of utility from calling-in immediately in $t = 0$ were already positive. To see this, consider the break-even level $\hat{r} := (g - \omega) + \frac{\theta}{\delta} \cdot (g - r^{ss} - \omega)$ for which $F(\hat{r}, 0) = 0$. For $g - \omega > r^{ss}$, the interval of postponement $(\underline{r}, \hat{r}]$ is non-empty with its length increasing in the speed of mean-reversion θ, in the net long-term period gain $g - r^{ss} - \omega$ and decreasing in the discount rate δ.

2.3 Optimal Timing under Uncertainty

At the time the calling-in option is exercised, its value must equal the expected present discounted value of cumulative future utility from time $T = 0$ on (*"value-matching condition"*), so $V(\underline{r}) = \mathbb{E}\left[\int_0^\infty (G(r_t) - \omega) \cdot \exp(-\delta t) dt \;\middle|\; r_0 = \underline{r}\right]$. The present discounted value of the non-stochastic portion of cumulative future dis-utility (i.e. welfare losses) is $\Omega := \mathbb{E}\left[\int_0^\infty \omega \cdot \exp(-\delta t) dt | r_0 = \underline{r}\right] = \frac{1}{\delta}\omega$. Thus, the value-matching condition with the *"termination payoff"* on the right-hand side of the equation in a canonical representation simply is

$$V(\underline{r}) = \mathbb{E}\left[\int_0^\infty (g - r_t) \cdot \exp(-\delta t) dt \;-\; \Omega \;\middle|\; r_0 = \underline{r}\right] \tag{9}$$

$$= \frac{1}{\delta} \cdot (g - r^{ss}) \;-\; \frac{1}{\delta + \theta} \cdot (\underline{r} - r^{ss}) \;-\; \Omega. \tag{10}$$

For $r \geq \underline{r}$, the value of the calling-in option for the instantaneous natural rate of interest r (dropping the time index) is given by the supremum over all stopping times T of the expected present discounted value of the termination payoff at time T:

$$V(r) = \sup_{\{T \geq 0\}} \mathbb{E}\left[\int_T^\infty (g - r_t) \cdot \exp(-\delta t) dt \;-\; \Omega \cdot \exp(-\delta T) \;\middle|\; r_0 = r\right]. \tag{11}$$

The optimal stopping problem (11) can be solved by applying dynamic programming methods as, for instance, described in Dixit and Pindyck (1994).[8] The Bellman equation as necessary condition the that must hold for the function V in the region $[\underline{r}, \infty)$, reads

$$\delta \cdot V dt = \mathbb{E}\left[dV\right]. \tag{12}$$

Applying the Itô formula to dV and rearranging (12) yields

$$\frac{1}{2}\sigma^2 V'' + \theta(r^{ss} - r)V' - \delta V = 0. \tag{13}$$

This is a second order ordinary differential equation with similar structure as the *Hermite Differential Equation*.[9] Letting w.l.o.g. $r^{ss} = 0$, the general solution of this equation is

$$V(r) = c_1 \cdot H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta} \cdot r}{\sigma}\right) \;+\; c_2 \cdot {}_1F_1\left(\frac{\delta}{2\theta}; \frac{1}{2}; \frac{\theta \cdot r^2}{\sigma^2}\right), \tag{14}$$

where $c_1, c_2 \in \mathbb{R}$ are constants, $H_\nu(z)$ is a Hermite function, and ${}_1F_1(a; b; x)$ is a confluent hypergeometric function of the first kind.[10]

Finding a particular solution of (13) is a free boundary problem with the domain of r being $[\underline{r}, \infty)$. Since $\underline{r}$ itself, c_1, and c_2 are to be determined, three boundary conditions

[8] For further details regarding the solution by dynamic programming that is used in the following see Dixit and Pindyck (1994, pp. 135-174).

[9] See, for instance, Abramowitz and Stegun (1964), Lebedev (1965), and Zwillinger (1997).

[10] See Lebedev (1965), http://functions.wolfram.com/HypergeometricFunctions/HermiteHGeneral/13/01/ and Weisstein (2017) for the general solution of the Hermite Differential Equation. See Appendix A for a detailed solution of the central bank's optimal stopping problem. See also Parlour and Walden (2009) who, although in another context, solve a similar ODE resulting from an OU process.

are needed. Two of these are the value-matching condition (9) and the *"smooth-pasting condition"*

$$V'(\underline{r}) = -\frac{1}{\delta + \theta}, \tag{15}$$

which requires V to have the same slope as the termination payoff function at $r = \underline{r}$.[11] As the value of the calling-in option in terms of cumulative discounted utility should be positive and decreasing in r, the third boundary condition is (with an abuse of notation)[12]

$$V(\infty) = 0. \tag{16}$$

Since $H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot r}{\sigma}\right) > 0$, and since $\lim_{r\to\pm\infty} {}_1F_1\left(\frac{\delta}{2\theta};\frac{1}{2};\frac{\theta\cdot r^2}{\sigma^2}\right) = \infty$ for $\delta, \theta, \sigma > 0$, the boundary condition (16) implies that $c_2 = 0$. Differentiating (14) w.r.t. r, inserting $V'(\underline{r})$ into the smooth-pasting condition (15), and solving for c_1 yields

$$c_1 = \frac{1}{\delta+\theta}\cdot\frac{\sqrt{\theta}\cdot\sigma}{2\delta}\cdot\frac{1}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r}}{\sigma}\right)}, \tag{17}$$

which is used to expand the value-matching condition (9). The result is

$$\frac{1}{\delta+\theta}\cdot\frac{\sqrt{\theta}\cdot\sigma}{2\delta}\cdot\frac{1}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r}}{\sigma}\right)}\cdot H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r}}{\sigma}\right) = \frac{g}{\delta} - \frac{1}{\delta+\theta}\cdot\underline{r} - \Omega, \tag{18}$$

which solves the optimal stopping problem and implicitly defines $\underline{r}$.[13]

3 Two-Country Setting

3.1 The Stopping Time Game between Two Central Banks

There are two countries, home and foreign, with two central banks (H in the home and F in the foreign country). The two countries (central banks) have their own currencies but at first are otherwise identical. Both central banks initially issue coin and paper money in

[11]For the concept and a discussion of *"smooth-pasting"* see Dixit and Pindyck (1994, pp. 130-132).

[12]Note that $r = \infty$ is a singular point of (13). The third condition that is used solving (13) actually is $\lim_{r\to\infty} V(r) = 0$. See Appendix A for a detailed derivation.

[13]Recall that w.l.o.g. $r^{ss} = 0$. See Appendix A for the general case of $r^{ss} \neq 0$.

the same denomination structure but each of them has the option to stop the issuance of high-denomination banknotes with immediate effect independently from its counterpart's action. The points in time when the calling-in options are exercised are denoted by T^H for the home and by T^F for the foreign central bank. After $t = \max\{T^H, T^F\}$, H and F will have the same denomination structures again. The domestic natural rates of interest r_t^H and r_t^F are identical, so the superscripts are dropped and $r_t^H = r_t^F = r_t$ which now simply is referred to as the world natural rate of interest.

Central banks' individual period utilities in periods $t < \min\{T^H, T^F\}$ are zero. H and F will stay fully symmetric in all regards forever if the calling-in moves are made simultaneously. In this case $U_t^H = U_t^F = G(r_t) - \omega$ for all $t \geq T^H = T^F$. However, if the calling-in moves are made sequentially central banks' period utilities will diverge during the interval $\Delta T := [\min\{T^H, T^F\}, \max\{T^H, T^F\})$. For $t \in \Delta T$, the first mover's utility is $U_t^{first} = G(r_t) - \omega + e^{first}$, the second mover's utility is $U_t^{second} = e^{second}$ for some $e^{first}, e^{second} \in \mathbb{R}$. Once $t \geq \max\{T^H, T^F\}$, symmetry is resolved and $U_t^H = U_t^F = G(r_t) - \omega$ forever.[14]

3.2 Strategic Interactions in a Deterministic Setting

Strategic behavior induced by a first- or second-mover advantage in a non-cooperative game between two central banks can result in inefficient timing decisions. The next scenario under perfect foresight is meant to provide some intuition on these issues and considers "strategic delay" when one central bank's calling-in move has a positive externality on the other central bank's utility. The structure of the externalities is $0 < e^{second} = -e^{first}$. The world natural rate converges to a low steady-state with $r_0 > g - \omega > r^{ss}$. As is usual in the literature on timing games the second-mover's ("follower") problem is solved first, assuming that one central bank has already made its calling-in move at the time the other one moves.[15] In what follows, without loss of generality, the second-mover's role is assigned to the foreign central bank and $T^F \geq T^H$.

[14]In another setting, period utilities could also depend on the number of central banks that already have made a calling-in move, for instance $U_t = G(r_t) - \omega + \chi$ for $t \geq \max\{T^H, T^F\}$.

[15]See, for instance, Dixit and Pindyck (1994, p. 310).

Under perfect foresight, all decisions are made in $t = 0$. In the absence of cooperation, the foreign central bank maximizes the objective function

$$F^F(r_0, T^{H^*}, T^F) = \int_{T^{H^*}}^{T^F} e^{second} \cdot \exp(-\delta t)dt \; + \; \int_{T^F}^{\infty} (g - r_t - \omega) \cdot \exp(-\delta t)dt \tag{19}$$

$$\begin{aligned} &= \frac{1}{\delta} e^{second} \cdot \Big(\exp(-\delta T^{H^*}) - \exp(-\delta T^F) \Big) \\ &+ \; \frac{1}{\delta}(g - r^{ss} - \omega) \cdot \exp(-\delta T^F) \\ &- \; \frac{1}{\delta + \theta}(r_0 - r^{ss}) \cdot \exp(-(\delta + \theta)T^F) \end{aligned} \tag{20}$$

by choosing some T^F given r_0 and T^{H^*} with the constraint $T^F \geq T^{H^*}$. The first-order condition for an interior maximum of (19) is

$$(g - r^{ss} - \omega - e^{second}) \cdot \exp(-\delta T^F) \;=\; (r_0 - r^{ss}) \cdot \exp(-(\delta + \theta)T^F). \tag{21}$$

The FOC (21) captures that, firstly, if T^{F^*} is an interior solution, it does not depend on the first-mover's timing, and, secondly, that the foreign central bank will free-ride on the home central bank's move forever if its advantage as a second-mover is sufficiently large, i.e. if $e^{second} \geq g - r^{ss} - \omega$ which then implies $T^{F^*} = \infty$. If the externality is relatively small, i.e. if $e^{second} < g - r^{ss} - \omega$, the foreign central bank does not completely avoid but only delay the calling-in move until some finite T^{F^*} for which the FOC (21) yields

$$T^{F^*} \;=\; \frac{1}{\theta} \cdot \ln \left(\frac{r_0 - r^{ss}}{g - r^{ss} - \omega - e^{second}} \right). \tag{22}$$

The externality on the foreign central bank's utility thus lowers the critical level $\underline{r}^F$ one-to-one down to $\underline{r}^F = g - \omega - e^{second}$.[16]

The home central bank anticipates the foreign central bank's reaction and thus its objective function, with the constraint $T^H \leq T^{F^*}$, is

$$F^H(r_0, T^{F^*}, T^H) = \int_{T^H}^{T^{F^*}} e^{first} \cdot \exp(-\delta t)dt \; + \; \int_{T^H}^{\infty} (g - r_t - \omega) \cdot \exp(-\delta t)dt. \tag{23}$$

[16] The next section analyzes the effect of uncertainty and whether uncertainty increases or decreases the impact of e on $\underline{r}$.

The first-order condition for an interior maximum of (23), using $e^{first} = -e^{second}$,

$$(g - r^{ss} - \omega - e^{second}) \cdot \exp(-\delta T^H) \;=\; (r_0 - r^{ss}) \cdot \exp(-(\delta + \theta)T^H), \tag{24}$$

yields

$$T^{H^*} \;=\; T^{F^*} \tag{25}$$

which implies a critical level $\underline{r^H} = \underline{r^F}$.

Clearly, the strategy profile $(T^{H^*}, T^{F^*}) = (\frac{1}{\theta}\ln\frac{r_0 - r^{ss}}{g - r^{ss} - \omega + e^{first}}, \frac{1}{\theta}\ln\frac{r_0 - r^{ss}}{g - r^{ss} - \omega - e^{second}})$ is an inefficient Nash equilibrium and both central banks could be better off by committing to the cooperative optimum $(T^H, T^F) = (\frac{1}{\theta}\ln\frac{r_0 - r^{ss}}{g - r^{ss} - \omega}, \frac{1}{\theta}\ln\frac{r_0 - r^{ss}}{g - r^{ss} - \omega})$.

3.3 Strategic Interactions under Uncertainty

To what extent do strategic incentives for the delay or the acceleration of a calling-in move depend on uncertainty about future states of the economy?

Consider an extended version of the optimal stopping problem with the associated Bellman equation (12) in a strategic setting as in section 3.2 but now under uncertainty about r_t. The home central bank has already made its calling-in move. The foreign central bank's problem is considered and thus the variable of interest is the critical level $\underline{r^F}$, i.e. the foreign central bank's stopping rule. For ease of notation, the present is $t = 0$ and time indices are dropped.

The foreign central bank receives the second mover's externality e^{second} until it makes its own calling-in move. So, the Bellman equation that must hold for its value function V^F is[17]

$$V^F(r) = \max\left\{\mathcal{V}^F(r),\; e(r)dt + (1 - \delta dt) \cdot V^F(r) + \mathbb{E}[dV^F]\right\} \tag{26}$$

[17] See Dixit and Pindyck (1994, p. 130) for further details and a discussion of a Bellman equation of this type.

where $\mathcal{V}^F$ is the follower's termination payoff defined as $\mathcal{V}^F(r) = \frac{1}{\delta}(g - r^{ss} - \omega) - \frac{1}{\delta+\theta}(r - r^{ss})$ (see (10) in the one-country setting at $r = \underline{r}$) and $e(r)$ is a function of r with

$$e(r) = \begin{cases} e^{second}, & \text{for } r \geq \underline{r^F} \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

For $r \in [\underline{r^F}, \infty)$, the Bellman equation (26) can be rearranged to

$$\delta \cdot V^F dt = e^{second} dt + \mathbb{E}[dV^F], \tag{28}$$

and, by using the Itô formula, written as an inhomogeneous differential equation

$$\frac{1}{2}\sigma^2 V^{F\prime\prime} + \theta(r^{ss} - r)V^{F\prime} - \delta V^F = -e^{second}. \tag{29}$$

This is just the Bellman equation (13) obtained in the one-country setting but extended by the inhomogeneous term $-e^{second}$. Three boundary conditions are needed to find a particular solution of (29). As the foreign central bank could simply choose never to make the calling-in move at all and hence to receive the externality e^{second} forever, the value of its calling-in option must be bounded from below by the perpetuity value of the externality. Thus, the right boundary condition for V^F is

$$V^F(\infty) = \frac{1}{\delta} e^{second}. \tag{30}$$

Therewith, the solution of (29) on the domain $[\underline{r^F}, \infty)$ must be

$$V^F(r) = c_1 \cdot H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta} \cdot r - \sqrt{\theta} \cdot r^{ss}}{\sigma}\right) + \frac{1}{\delta} e^{second}, \tag{31}$$

where $c_1 \in \mathbb{R}$ is a constant and $H_\nu(z)$ a Hermite function. A particular solution of (29) then can be obtained by using the value-matching and smooth-pasting conditions

$$V^F(\underline{r^F}) = \frac{1}{\delta} \cdot (g - r^{ss} - \omega) \; - \; \frac{1}{\delta+\theta} \cdot \left(\underline{r^F} - r^{ss}\right), \tag{32}$$

and

$$V^{F\prime}(\underline{r^F}) = -\frac{1}{\delta+\theta}, \tag{33}$$

which correspond to the respective conditions in the one-country setting: The reason that e^{second} is irrelevant for the two left boundary conditions (32) and (33) is that it is lost at the instant the foreign central bank makes its calling-in move, i.e. when r_t hits $\underline{r^F}$ the first time. Like in section 2.3, the smooth-pasting condition can be used to replace c_1 in an expanded form of the value-matching condition equation (32), i.e.

$$c_1 \;=\; \frac{1}{\delta+\theta}\cdot\frac{\sqrt{\theta}\cdot\sigma}{2\delta}\cdot\frac{1}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r^F}}{\sigma}-\frac{\sqrt{\theta}\cdot r^{ss}}{\sigma}\right)}, \tag{34}$$

can be used to obtain an implicit definition of $\underline{r^F}$ in

$$\frac{1}{\delta+\theta}\cdot\frac{\sqrt{\theta}\cdot\sigma}{2\delta}\cdot\frac{H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r^F}}{\sigma}-\frac{\sqrt{\theta}\cdot r^{ss}}{\sigma}\right)}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r^F}}{\sigma}-\frac{\sqrt{\theta}\cdot r^{ss}}{\sigma}\right)}+\frac{1}{\delta}e^{second}=\frac{1}{\delta}(g-r^{ss}-\omega)-\frac{1}{\delta+\theta}(\underline{r^F}-r^{ss}). \tag{35}$$

$$\frac{1}{\delta+\theta}\cdot\frac{\sqrt{\theta}\cdot\sigma}{2\delta}\cdot\frac{H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r^F}}{\sigma}-\frac{\sqrt{\theta}\cdot r^{ss}}{\sigma}\right)}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r^F}}{\sigma}-\frac{\sqrt{\theta}\cdot r^{ss}}{\sigma}\right)}=\frac{1}{\delta}(g-r^{ss}-\omega-e^{second})-\frac{1}{\delta+\theta}(\underline{r^F}-r^{ss}). \tag{36}$$

4 Concluding Remarks

The first part of our analysis has shown how the optimal timing of calling-in large-denomination banknotes depends on the properties of the stochastic process that governs the path of the natural rate of interest. The model with the mean-reverting Ornstein-Uhlenbeck process we chose can easily be extended, for instance, by assuming that the natural rate follows a jump diffusion. This could account for the hypothesis that large-scale economic or financial crises can lead to exceptional drops in the natural rate in their immediate aftermath, as indicated by the empirical findings of Holston, Laubach, and

Williams (2017). Further analysis along this dimension will be interesting as we have shown that the stochastic properties of the natural rate determine to what extent a *'wait-and-see-'*value of the option to call-in large notes will induce central banks to postpone any desired changes in the denominational structure of their currency to a future date.

In the second part of our analysis we have shown how strategic interactions induced by externalities to unilateral calling-in moves will lead to inefficient timing decisions and to what extent uncertainty about the future path of the natural rate does affect the underlying strategic considerations. One of various scenarios that we have left out for future analysis is one where fully symmetric central banks will move sequentially, and not simultaneously like discussed in Section 3. Such a scenario where symmetric players chose asymmetric strategies is analyzed, for instance (but there in an industrial organization context), in Reinganum (1981). Further analysis of game theoretic aspects in different scenarios will contribute to the evaluation of potential welfare gains from the international coordination of changes in the denominational structures of major global currencies.

Appendices

A Detailed Solution of the Optimal Stopping Problem in the One-Country Setting

This appendix provides a detailed solution of the central bank's optimal stopping problem (11) in the one-country setting. The problem is solved by following the procedures and using the dynamic programming methods described in Dixit and Pindyck (1994). As far as possible, the (shorthand) notation proposed therein is used as well. For instance, Dixit and Pindyck (1994) describe how to solve an optimal stopping problem where the underlying process is geometric Brownian motion (ibid., pp. 135-160) and how to solve a problem with a mean-reverting process that - in contrast to (37) - has an absorbing state (see ibid., pp. 161-167). The complementary reference for some basic methods of Itô calculus used here is Øksendal (2003), the reference for special functions and differential equations is Lebedev (1965).

A.1 Derivation of the Bellman equation written as ODE

Recall the mean-reverting Ornstein-Uhlenbeck process describing the evolution of the natural rate of interest

$$dr_t = \theta(r^{ss} - r_t)dt + \sigma dB_t, \tag{37}$$

where B_t is Brownian motion and $\theta \in \mathbb{R}_{>0}$, $r^{ss} \in \mathbb{R}$, and $\sigma \in \mathbb{R}_{\geq 0}$ are known constants.

In what follows, we also need

$$(dr_t)^2 = (dr_t) \cdot (dr_t) \tag{38}$$

$$= \theta^2(r^{ss} - r_t)^2(dt)^2 + 2 \cdot \theta(r^{ss} - r_t)dt \cdot \sigma dB_t + \sigma^2(dB_t)^2 \tag{39}$$

$$= \sigma^2 dt, \tag{40}$$

which is obtained by using the rules $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$ and $dB_t \cdot dB_t = dt$ given in Øksendal (2003, p. 45).

Let the function $V : \mathbb{R} \to \mathbb{R}$ be twice differentiable. The function $V : r \mapsto V(r)$ is independent of time and measures the value of the calling-in option dependent on the natural rate of interest only, so the time index is dropped and $r = r_t$.[18] Applying the Itô formula, using (37), (40), and the notation $V' = \frac{\partial V}{\partial r}$ respectively $V'' = \frac{\partial^2 V}{\partial r^2}$, it is[19]

$$dV = \frac{\partial V}{\partial r}dr + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}(dr)^2 \tag{41}$$

$$= V' \cdot (\theta(r^{ss} - r)dt + \sigma dB_t) + \frac{1}{2}V'' \cdot \sigma^2 dt, \tag{42}$$

with which we obtain[20]

$$\mathbb{E}[dV] = \theta(r^{ss} - r) \cdot V' dt + \frac{1}{2}\sigma^2 \cdot V'' dt. \tag{43}$$

[18]The properties of value functions in similar infinite time horizon optimal stopping settings are discussed in Dixit and Pindyck (1994, p. 107).

[19]The Itô formula in the shorthand notation used here is given in Dixit and Pindyck (1994, p. 80). A complete discussion of this formula in a general context is given in Øksendal (2003, p. 44).

[20]See Dixit and Pindyck (1994, pp. 140-141) for a similar problem. However, the stochastic process that governs their state variable has - in contrast to (37) - an absorbing state.

Now, the optimality condition implicitly defining V for $r \in [\underline{r}, \infty)$ is the Bellman equation[21]

$$\delta \cdot V dt = \mathbb{E}\left[dV\right], \tag{44}$$

which, using (43), can be written as homogeneous ordinary differential equation

$$\frac{1}{2}\sigma^2 V'' + \theta(r^{ss} - r)V' - \delta V = 0. \tag{45}$$

A.2 Transformation of the Bellman equation into a Canonical Form

The algorithm for the transformation of an arbitrary ordinary differential equation into a canonical form used in the following is taken from Dixit and Pindyck (1994, p. 163). The initial substitution $z(r)$ used to transform (45) has been obtained with the computer algebra system *Mathematica*.[22]

Let $z : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $r \mapsto \frac{\sqrt{\theta}}{\sigma} r - \frac{\sqrt{\theta}}{\sigma} r^{ss}$ and $z'(r) = \frac{\sqrt{\theta}}{\sigma}$. Moreover, let $w : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $w : z \mapsto w(z)$. Analogously to the procedure described in Dixit and Pindyck (1994, p. 163), we make the substitution $V(r) = w(z)$ with $V'(r) = w'(z(r)) \cdot z'(r) = \frac{\sqrt{\theta}}{\sigma} w'(z)$ and $V''(r) = \frac{\theta}{\sigma^2} w''(z)$. Therewith, and with $r = \frac{\sigma}{\sqrt{\theta}} z + r^{ss}$, equation (45) can be transformed into

$$\frac{1}{2}\sigma^2 \frac{\theta}{\sigma^2} w''(z) + \theta(r^{ss} - (\frac{\sigma}{\sqrt{\theta}} z + r^{ss}))\frac{\sqrt{\theta}}{\sigma} w'(z) - \delta w(z) = 0, \tag{46}$$

which can be simplified to

$$w'' - 2zw' + 2\nu w = 0, \tag{47}$$

where $\nu := -\frac{\delta}{\theta} < 0$. This differential equation is discussed, for instance, in great detail in Lebedev (1965, pp. 283-300).[23]

[21]See the related Bellman equation in Dixit and Pindyck (1994, p. 140).
[22]See Wolfram Research, Inc. (2015).
[23]As discussed in Lebedev (1965, p. 284), for $\nu \in \mathbb{N}_0$, equation (47) is related to the Hermite polynomials which are applied, i. a., in quantum mechanics to describe a linear harmonic oscillator (see ibid., p. 297).

A.3 General Solution of the Bellman Equation

The general solution of (47) can be found in Lebedev (1965, p. 286, equation (10.2.17)) or with *Mathematica*. According to Lebedev, it reads

$$w = MH_{\nu}(z) + N\exp(z^2)H_{-\nu-1}(\mathbb{i}z), \tag{48}$$

where $M, N \in \mathbb{C}$ are constants, $\mathbb{i}^2 = -1$, and $H_{\nu}(z)$ is a Hermite function (as defined, for instance, in Lebedev (1965, p. 285)).

Resubstituting with $w = V$, $z = \frac{\sqrt{\theta}}{\sigma}r - \frac{\sqrt{\theta}}{\sigma}r^{ss}$, and $\nu = -\frac{\delta}{\theta}$, the general solution of the Bellman equation (45) is

$$\begin{aligned} V(r) = {} & c_1 H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma}r - \frac{\sqrt{\theta}}{\sigma}r^{ss}\right) \\ & + c_2 \exp\left(\left(\frac{\sqrt{\theta}}{\sigma}r - \frac{\sqrt{\theta}}{\sigma}r^{ss}\right)^2\right) H_{\frac{\delta}{\theta}-1}\left(\mathbb{i}\left(\frac{\sqrt{\theta}}{\sigma}r - \frac{\sqrt{\theta}}{\sigma}r^{ss}\right)\right), \end{aligned} \tag{49}$$

where $c_1, c_2 \in \mathbb{C}$ are constants.

A.4 Particular Solution of the Bellman Equation and Derivation of The Value Function

The economic nature of the central bank's optimal stopping problem requires that any particular solution of (45) must satisfy the conditions

$$V \geq 0, \quad \text{and} \tag{50}$$

$$\lim_{r\to\infty} V = 0 \quad (\text{"right boundary condition"}). \tag{51}$$

Proposition 1. *Consider the general solution of the Bellman equation with* $V(r)$ *as given in* (49) *and let* $c_2 = 0$*. Then,*

$$V(r) = c_1 H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}}{\sigma}r - \frac{\sqrt{\theta}}{\sigma}r^{ss}\right) \tag{52}$$

satisfies conditions (50) *and* (51) *for all* $c_1 \in \mathbb{R}_{>0}$ *and* $\frac{\delta}{\theta} > 0$*.*

Proof. Consider the integral representation of the Hermite function for Re $\nu < 0$ as given in Lebedev (1965, p. 290, equation (10.5.1)) with

$$H_\nu(z) = \frac{1}{2\Gamma(-\nu)} \int_0^\infty \exp(-s - 2z\sqrt{s}) s^{-\frac{1}{2}\nu - 1} ds, \tag{53}$$

where $\Gamma(\cdot)$ is the Gamma function as defined in Lebedev (1965, p. 1). With $\nu = -\frac{\delta}{\theta} \in \mathbb{R}_{<0}$ and therefore $\Gamma(-\nu) \in \mathbb{R}_{>0}$ it becomes immediately clear that $H_\nu(z) > 0$ for all $z \in \mathbb{R}$ which proves the first part of the proposition. A direct implication of this property is the monotonicity of V since $H'_\nu(z) = 2\nu H_{\nu-1}(z)$ (see Lebedev (1965, p. 289, equation (10.4.4))). To prove the second part of the proposition, consider the asymptotic representation of $H_\nu(z)$ for large $|z|$ as given in Lebedev (1965, p. 292, equation (10.6.6)) for the special case of $z \in \mathbb{R}$ with

$$H_\nu(z) = (2z)^\nu \left[\sum_{k=0}^{n} \frac{(-1)^k}{k!} (-\nu)_{2k} (2z)^{-2k} + O(|z|^{-2n-2}) \right], \tag{54}$$

where $(-\nu)_0 = 1$ and $(-\nu)_{2k} = \frac{\Gamma(-\nu+2k)}{\Gamma(-\nu)} = (-\nu)(-\nu+1)\cdots(-\nu+2k-1)$. The second part of the proposition follows directly from (54) if $\nu < 0$.[24] □

Now, we can derive a particular solution of the Bellman equation starting with (52) and determining c_1 and $\underline{r}$ to solve the central bank's decision problem.[25] The *"smooth-pasting condition"*

$$V'(\underline{r}) = -\frac{1}{\delta + \theta} \tag{55}$$

can be used to obtain

$$c_1 = \frac{1}{\delta+\theta} \cdot \frac{\sqrt{\theta} \cdot \sigma}{2\delta} \cdot \frac{1}{H_{-1-\frac{\delta}{\theta}} \left(\frac{\sqrt{\theta} \cdot \underline{r}}{\sigma} - \frac{\sqrt{\theta} \cdot r^{ss}}{\sigma} \right)}, \tag{56}$$

which, together with the *"value-matching condition"*

$$V(\underline{r}) = \frac{1}{\delta} \cdot (g - r^{ss} - \omega) - \frac{1}{\delta+\theta} \cdot (\underline{r} - r^{ss}), \tag{57}$$

[24] To see this, let $n = 0$ and consider $H_\nu(z) = (2z)^\nu \left[1 + O(|z|^{-2})\right]$ for $\nu < 0$.

[25] For the concept and a discussion *"smooth-pasting"* see Dixit and Pindyck (1994, pp. 130-132).

can be used to obtain an implicit definition of $\underline{r}$ in

$$\frac{1}{\delta+\theta}\cdot\frac{\sqrt{\theta}\cdot\sigma}{2\delta}\cdot\frac{H_{-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r}}{\sigma}-\frac{\sqrt{\theta}\cdot r^{ss}}{\sigma}\right)}{H_{-1-\frac{\delta}{\theta}}\left(\frac{\sqrt{\theta}\cdot\underline{r}}{\sigma}-\frac{\sqrt{\theta}\cdot r^{ss}}{\sigma}\right)}=\frac{1}{\delta}(g-r^{ss}-\omega)-\frac{1}{\delta+\theta}(\underline{r}-r^{ss}). \tag{58}$$

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